Definition 1.1

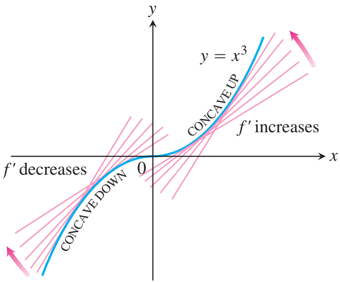
The graph of a differentiable function $y = f(x)$ is:

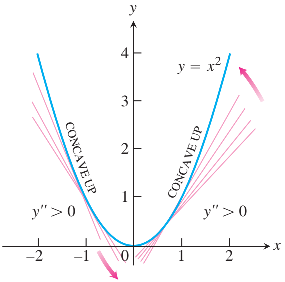
- **concave up** on an open interval I if f' is increasing on I
- **concave down** on an open interval I if f' is decreasing on I

Theorem 1.2

Let $y = f(x)$ be twice-differentiable on an open interval I.

- *If $f'' > 0$ on I, the graph of f over I is concave up*
- *If $f'' < 0$ on I, the graph of f over I is concave down.*


y
x
$y = x^3$
CONCAVE UP
f' increases
f' decreases
0
CONCAVE DOWN


y
x
$y = x^2$
4
3
2
1
CONCAVE UP
CONCAVE UP
$y'' > 0$
$y'' > 0$
–2
–1
0
1
2

Definition 1.3

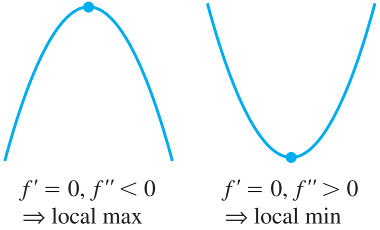
A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**

Note: To find the points of inflection, we need to find the zeros of f'' and points where f'' is not defined. Then we need to check the concavity.

Example 1.4

The curve $y = x^4$ has no inflection point at $x = 0$. Even though $y'' = 12x^2$ is zero there, it does not change sign

Note: An inflection point may not exist where $y'' = 0$



Theorem 1.5

Second derivative test for local maximums and minimums: suppose that f'' is continuous on an open interval that contains $x = c$.

- *If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$*
- *If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$*
- *If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.*

Example 1.6

Find and classify the stationary points and the points of inflection of

$$f(x) = 4x^3 - 21x^2 + 18x + 6$$

$f'(x) = 12x^2 - 42x + 18$
When $f'(x) = 0$, we have:

$$12x^2 - 42x + 18 = 0 \Rightarrow 2x^2 - 7x + 3 = 0 \Rightarrow (2x - 1)(x - 3) = 0$$

So The stationary points are: $x = 1/2$ and $x = 3$
$f''(x) = 24x - 42$ so

$$f''(1/2) = 24(1/2) - 42 = 12 - 42 < 0$$

which means $x = 1/2$ is a local max. Also as

$$f''(3) = 24(3) - 42 = 72 - 42 > 0$$

$x = 3$ is a local min.