MA140-Engineering Calculus

Lecture 10

September 29, 2017

Exercises

Prove that

- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(e^x) = e^x$

Basic Rules of Differentiation

(1) *Derivative of a Constant Function:* If f has the constant value $f(x) = c$, then:

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0$$

(2) *Power Rule for Positive Integers:* If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

(3) *Constant Multiple Rule:* If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

(4) *Derivative Sum Rule:* If u and v are differentiable functions of x, then their sum $u+v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

(5) *Derivative Product Rule:* If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

(6) *Derivative Quotient Rule:* If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}(\frac{u}{v}) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example 1.1

Suppose that $f(x) = -5x^3 + 3x^2 - 9x + 7$, then find:

(a) The derivative of $f(x)$

(b) The slope of the tangent line at $x = 2$

(c) The equation of the tangent at $x = 2$

Note: The equation of a line with slope m and a point (x_1, y_1) on the line is:

$$y - y_1 = m(x - x_1)$$

(a):

$$f'(x) = -15x^2 + 6x - 9$$

(b): The slope of the tangent line at $x = 2$ is $f'(2)$

$$f'(2) = -15(2)^2 + 6(2) - 9 = -15(4) + 12 - 9 = -60 + 12 - 9 = -57$$

(c): The y coordinate at $x = 2$ is

$$f(2) = -5(2)^3 + 3(2)^2 - 9(2) + 7 = -5(8) + 3(4) - 18 + 7$$

$$= -40 + 12 - 18 + 7 = -39$$

So $(2, -39)$ is a point on the tangent line and the slope of the line is -57 so the equation of the line is:

$$y - y_1 = m(x - x_1) = y + 39 = -57(x - 2)$$

Example 1.2

Use the rules to show that

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}(\frac{\sin x}{\cos x})$$

Using the quotient rule, we will get

$$\frac{d}{dx}(\frac{\sin x}{\cos x}) = \frac{(\cos x)[\frac{d}{dx}(\sin x)] - (\sin x)[\frac{d}{dx}(\cos x)]}{(\cos^2 x)}$$

Exercises

Use the rules to show that:

- $$\frac{d}{dx}(\sec x) = \sec(x) \cdot \tan(x)$$
- $$\frac{d}{dx}(\csc(x)) = -\csc(x) \cdot \cot(x)$$

  Hint: $\csc(x) = 1/\sin(x) \quad \textit{and} \quad \cot(x) = 1/\tan(x)$
- $$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$$

Definition 1.3

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

In another notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 1.4

If $y = (x^3 + 4x^4 + 7)^{99}$, find $\frac{dy}{dx}$

Let $u = x^3 + 4x^4 + 7$, we can write y as $y = u^{99}$, then by chain rule we have:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 99u^{98}[3x^2 + 16x^3],$$

so

$$\frac{dy}{dx} = 99(x^3 + 4x^4 + 7)^{98}(3x^2 + 16x^3)$$

Example 1.5

If $y = \dfrac{1000}{(x^4 + 2x^2 + 8)^{40}}$, find $\frac{dy}{dx}$

$y = 1000(x^4 + 2x^2 + 8)^{-40}$.
Let $u = x^4 + 2x^2 + 8$, so we can write $y = 1000u^{-40}$
The Chain rule is:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Now:

$$\frac{dy}{du} = (1000)(-40)u^{-41}$$

$$\frac{du}{dx} = 4x^3 + 4x$$

so

$$\frac{dy}{dx} = -40000u^{-41}[4x^3 + 4x]$$

then

$$\frac{dy}{dx} = \frac{-40000}{u^{41}}(4x^3 + 4x)$$

or

$$\frac{dy}{dx} = \frac{-40000(4x^3 + 4x)}{x^4 + 2x^2 + 8}$$

Note: Sometimes it is useful to involve a second (or more) intermediate function

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

Example 1.6

Find $\frac{dy}{dx}$, when

$$y = \sin^4(x^5 + 7)$$

Let $u = \sin(x^5 + 7)$ and let $v = x^5 + 7$
so the chain rule gives

$$\frac{d\sin^4(x^5+7)}{dx} = \frac{d\sin^4(x^5+7)}{d\sin(x^5+7)} \cdot \frac{d\sin(x^5+7)}{d(x^5+7)} \cdot \frac{d(x^5+7)}{dx}$$

$$\left[4\sin^3(x^5+7)\right] \cdot \left[\cos(x^5+7)\right] \cdot \left[5x^4\right] = 20x^4 \cdot \sin^3(x^5+7) \cdot \cos(x^5+7)$$

Exercises

Find $\frac{dy}{dx}$, if:

- $$y = x^2 e^{\sin x}$$
- $$y = \tan^3[\sin^2(x^4)]$$