MA140-Engineering Calculus

Lecture 35

November 22, 2017

First Order Linear Differential Equations (The Integrating Factor Method):

Definition 1.1

A D.E. of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

is called a **first order linear D.E.**, where $P(x)$ and $Q(x)$ are functions of x.

We can find the general solution of this D.E. as follows:

1.) Find the **integrating factor**

$$e^{\int P(x)dx}$$

2.) Multiply the D.E. by the integrating factor (I.F.) to get:

$$e^{\int P(x)dx}\left(\frac{dy}{dx} + P(x)y\right) = e^{\int P(x)dx}Q(x) \quad (\star)$$

3.) Note that the L.H.S. of $(\star)$ equals:

$$\frac{d}{dx}(ye^{\int P(x)dx})$$

So $(\star)$ becomes:

$$d(ye^{\int P(x)dx}) = e^{\int P(x)dx}Q(x)dx$$

4.) Integrate both sides and solve for y.
This algorithm is called the **integrating factor method**.

Example 1.2

Solve the first order linear differential equation

$$\frac{dy}{dx} - 3y = 0$$

using the integrating factor method.

1.)

$$I.F. = e^{\int P(x)dx} = e^{\int -3dx} = e^{-3x}$$

(note that we do not include the arbitrary constant C).
2.) Multiply the D.E. by the I.F. to get

$$e^{-3x}(\frac{dy}{dx} - 3y) = 0e^{-3x} \quad (\star)$$

3.) Recall that the L.H.S. of $(\star)$ equals

$$\frac{d}{dx}(ye^{-3x})$$

therefore

$$\frac{d}{dx}(ye^{-3x}) = 0 \Rightarrow d(ye^{-3x}) = 0dx$$

4.) Integrate and solve for y:

$$\int d(ye^{-3x}) = \int 0dx \Rightarrow ye^{-3x} = 0 + C \Rightarrow y = Ce^{3x}$$

Example 1.3

Solve the first order linear differential equation

$$\frac{dy}{dx} - y = e^x$$

using the integrating factor method.

1.)

$$I.F. = e^{\int P(x)dx} = e^{\int -1dx} = e^{-x}$$

(note that we do not include the arbitrary constant C).

2.) Multiply the D.E. by the I.F. to get

$$e^{-x}(\frac{dy}{dx} - y) = e^x \cdot e^{-x} \quad (\star)$$

3.) Recall that the L.H.S. of $(\star)$ equals

$$\frac{d}{dx}(ye^{-x})$$

therefore

$$\frac{d}{dx}(ye^{-x}) = 1 \Rightarrow d(ye^{-x}) = 1dx$$

4.) Integrate and solve for y:

$$\int d(ye^{-x}) = \int 1dx \Rightarrow ye^{-x} = x + C \Rightarrow y = xe^{x} + c$$

Last year exam Solutions:

Question $1.(a).i$:

$$\lim_{x\to 1}\frac{x-1}{(\sqrt{x}-1)(x+2)} = \lim_{x\to 1}\frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(x+2)}$$

$$= \lim_{x\to 1}\frac{\sqrt{x}+1}{x+2} = \frac{2}{3}$$

Question $1.(a).ii$:

we apply L'Hôpital's Rule:

$$\lim_{\theta\to 0}\frac{6\sin\theta}{\theta+2\tan\theta} \overset{\text{H}}{=} \lim_{\theta\to 0}\frac{6\cos\theta}{1+2\sec^2\theta} = \frac{6}{3} = 2$$

Question $2.(a).i$:

The derivative of the function $f(x)$ with respect to the variable x is the function f' or $\frac{df}{dx}$ whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Question $2.(a).ii$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 + (x+h) - (x^2 + x)}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + h^2 + 2xh + x + h - x^2 - x}{h} = \lim_{h \to 0} \frac{h^2 + 2xh + h}{h}$$

$$= \lim_{h \to 0} (h + 2x + 1) = 2x + 1$$

Question $2.(b).i$:

$$f(x) = e^{\cos x^2} \sin x$$

$$\Rightarrow f'(x) = (e^{\cos x^2} \sin x)' = (e^{\cos x^2})' \sin x + e^{\cos x^2} (\sin x)' \quad (\star)$$

To differentiate $e^{\cos x^2}$ we use the chain rule:
Let $y = e^{\cos x^2} = e^u$, where $u = \cos x^2$, then by chain rule we have:

$$y' = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot (\cos x^2)'$$

To differentiate $\cos x^2$ we use the chain rule again:
$u = \cos x^2 = \cos v$, where $v = x^2$, then by chain rule we have:

$$u' = \frac{du}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx} = (-\sin v)(2x) = (-\sin x^2)(2x)$$

So

$$(e^{\cos x^2})' = y' = e^u \cdot (-\sin x^2)(2x) = (e^{\cos x^2}) \cdot (-\sin x^2)(2x)$$

Therefore

$$(\star) = f'(x) = (e^{\cos x^2}) \cdot (-\sin x^2)(2x) \sin x + e^{\cos x^2} (\cos x)$$

Question $2.(b).iii$:

$$y = (\cos x)^x$$

$$\Rightarrow \ln y = \ln(\cos x)^x \Rightarrow \ln y = x \ln \cos x$$

$$\Rightarrow (\ln y)' = (x \ln \cos x)' \Rightarrow \frac{y'}{y} = 1 \cdot \ln \cos x + x \cdot \frac{(\cos x)'}{\cos x}$$

$$\Rightarrow \frac{y'}{y} = \ln \cos x - \frac{x \sin x}{\cos x}$$

$$\Rightarrow y' = y\left[\ln \cos x - \frac{x \sin x}{\cos x}\right] = (\cos x)^x \left[\ln \cos x - \frac{x \sin x}{\cos x}\right]$$

Semester I Examinations 2016/2017

Exam Codes	1BE1,1BG1,1BLE1,1BM1,1BP1,1BSE1,1EG1.
Exams	First Year Engineering.
Module Code	MA 140.
Modules	Engineering Calculus.
Paper No	1.
Repeat Paper	No.
External Examiner	Prof. M. Lawson,
Internal Examiners	Prof. G. Ellis, Dr P. Browne.
Instructions	Answer Any Four (4) questions. Full marks for Four (4) correct solutions. **[25 marks for each question]**
Duration	TWO HOURS.
No. of Pages	5 pages.
Discipline	Mathematics.

Requirements:

Release in Exam venue	Yes ☑	No ☐
MCQ	Yes ☐	No ☑
Statistical / Log Tables	Yes ☑	No ☐

1. (a) Evaluate the following limits,

i.

$$\lim_{x \to 1} \frac{x-1}{(\sqrt{x}-1)(x+2)},$$

[2]

ii.

$$\lim_{\theta \to 0} \frac{6 \sin \theta}{\theta + 2 \tan \theta}.$$

[3]

(b) i. State the Intermediate Value Theorem.

[4]

ii. Use the Intermediate value theorem to show that the equation

$$8x^3 - x^2 + 4x - 1 = 0,$$

has a root, $c \in [-1, 1]$.

[5]

(c) Let,

$$h(x) := \begin{cases} -1 & \text{for } x \leq 0, \\ x + 2 & \text{for } 0 < x < 1, \\ x^2 & \text{for } x \geq 1. \end{cases}$$

i. Making use of the empty and full circle notation, sketch the graph of $h(x)$.

[3]

ii. What does it mean for a function to be *continuous at a point* c ?

[4]

iii. Compute $\lim_{x \to 1^-} h(x)$ and $\lim_{x \to 1^+} h(x)$. Explain whether the function is continuous at $x = 1$ or not ?

[4]

p.t.o

2. (a) i. Give the definition of the *derivative of a function* $f(x)$.

[2]

ii. Differentiate the function, $f(x) = x^2 + x$, **from first principles**.

[3]

(b) Differentiate the following functions with respect to x (you do not need to use first principles).

i. $f(x) = e^{\cos x^2} \sin x$.

[5]

ii. $h(x) = \dfrac{x^3 + 2x}{\sqrt{x}}$.

[5]

iii. $y(x) = (\cos x)^x$.

[5]

(c) Your battleship sails along the line: $L : y = 3x + 1$. As captain you wish to determine how close you will come to the grid coordinate $(1, 1)$. Find the point on L that has minimal distance to the point $(1, 1)$.

[5]

p.t.o

3. (a) Let

$$f(x) = \frac{1}{1 + e^{3x}}.$$

i. Find all asymptotes of the graph of $f(x)$.

[4]

ii. Determine the interval on which $f(x)$ is decreasing.

[4]

iii. Determine the interval on which $f(x)$ is increasing.

[4]

iv. Show how to and find all point(s) of inflection for the graph of $f(x)$.

[4]

v. Sketch the graph of $f(x)$.

[4]

(b) Sketch the region below the graph of the curve:

$$y(x) = \frac{1}{2x - 1},$$

and above the $x-$axis for $x \in [3, 4]$.
Now compute the volume of the solid obtained by rotating this region about the $x-$axis.

[5]

p.t.o

4. (a) State the *Fundamental Theorem of Calculus*, and use it to differentiate:

$$\int_1^{x^4} \sin t^3 dt.$$

[10]

(b) Compute the following indefinite integrals:

i. $\int 2x\sqrt{1+x^2}\ dx$.

[5]

ii. $\int x \cos x\ dx$.

[5]

iii. $\int \ln(x)\ dx$.

[5]

5. (a) Define the term *arc length*, and show that the arc length of a curve $y(x)$ from $x = a$ to $x = b$ is given by:

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

[10]

(b) Define the term *curvature*, and show that the curvature of a circle of radius λ is $\frac{1}{\lambda}$.

[9]

(c) Staff moral over time in a department in a university is given by the function $M(t)$, where t is time. Over the past few years it has been noticed that this function obeys the following differential equation

$$\frac{d}{dt}M(t) = kM(t),$$

where $k < 0$. Find the moral function $M(t)$.

[6]

END OF EXAM.