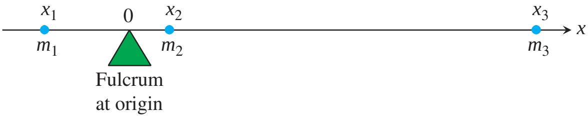
MA140-Engineering Calculus

Lecture 31

November 14, 2017

Centers of Mass:
Masses Along a Line:



Each mass m_k exerts a downward force $m_k g$.
Each of these forces has a tendency to turn the axis about the origin This turning effect is called a torque.
The torque of each mass is:

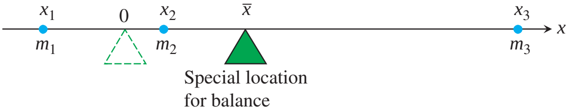
$$m_k g x_k$$

$$System \quad torque = m_1 g x_1 + m_2 g x_2 + m_3 g x_3$$

$$= g \cdot (m_1 x_1 + m_2 x_2 + m_3 x_3)$$

The number $m_1 x_1 + m_2 x_2 + m_3 x_3$ is called the moment of the system about the origin.

We usually want to know where to place the fulcrum to make the system balance, that is, at what point $\bar{x}$ to place it to make the torques add to zero.



The torque of each mass about the fulcrum in this special location is:

$$(x_k - \bar{x})m_k g$$

So the torque of the new system is

$$\sum_{k=1}^{3}(x_k - \bar{x})m_k g$$

Then:

$$\sum_{k=1}^{3}(x_k - \bar{x})m_k g = 0 \Rightarrow g\sum_{k=1}^{3}(x_k - \bar{x})m_k = 0$$

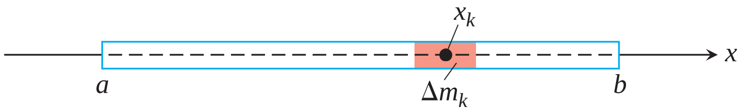
$$\Rightarrow \sum_{k=1}^{3}(m_k x_k - m_k\bar{x}) = 0 \Rightarrow \sum_{k=1}^{3} m_k x_k - \sum_{k=1}^{3} m_k \bar{x} = 0$$

$$\Rightarrow \sum_{k=1}^{3} m_k x_k = \bar{x}\sum_{k=1}^{3} m_k \Rightarrow \bar{x} = \frac{\sum_{k=1}^{3} m_k x_k}{\sum_{k=1}^{3} m_k}$$

This last equation tells us to find $\bar{x}$ by dividing the system's moment about the origin by the system's total mass:

$$\Rightarrow \bar{x} = \frac{\sum_{k=1}^{3} m_k x_k}{\sum_{k=1}^{3} m_k} = \frac{system\ \ moment\ \ about\ \ origin}{system\ \ mass}$$

Wires and Thin strips:



$$\bar{x} \cong \frac{system\ \ moment}{system\ \ mass}$$

The system mass is

$$\sum_{k=1}^{n} \Delta m_k$$

the moment of each piece $x_k \Delta m_k$, so

$$System\ \ moment \cong \sum_{k=1}^{n} x_k \Delta m_k$$

if the density of the strip at x_k is $\delta(x_k)$, expressed in terms of mass per unit length and if δ is continuous, then Δm_k is approximately equal to $\delta(x_k)\Delta x_k$:

$$\Delta m_k \cong \delta(x_k)\Delta x$$

Therefore:

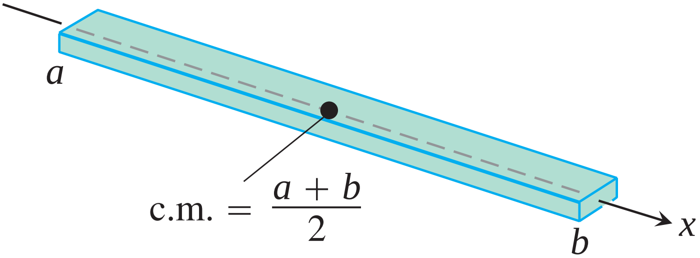
$$\bar{x} \cong \frac{system\ \ moment}{system\ \ mass} \cong \frac{\sum_{k=1}^{n} x_k \Delta m_k}{\sum_{k=1}^{n} \Delta m_k} \cong \frac{\sum_{k=1}^{n} x_k \delta(x_k) \Delta x}{\sum_{k=1}^{n} \delta(x_k) \Delta x}$$

The sums is a Riemann sum so when $\Delta x \to 0$ or $n \to \infty$, then

$$\bar{x} = \frac{\int_a^b x\delta(x)dx}{\int_a^b \delta(x)dx}$$

Example 1.1

Show that the center of mass of a straight, thin strip or rod of constant density lies halfway between its two ends.



We model the strip as a portion of the x-axis from $x = a$ to $x = b$. We know that

$$\bar{x} = \frac{\int_a^b x\delta(x)dx}{\int_a^b \delta(x)dx}$$

the density is constant so $\delta(x) = \delta$. The numerator is:

$$\int_a^b x\delta dx = \delta \int_a^b x dx = \delta \frac{1}{2}x^2\Big]_a^b = \frac{\delta}{2}(b^2 - a^2)$$

The denominator is:

$$\int_a^b \delta dx = \delta \int_a^b dx = \delta x\Big]_a^b = \delta(b - a)$$

So:

$$\bar{x} = \frac{\int_a^b x\delta(x)dx}{\int_a^b \delta(x)dx} = \frac{a+b}{2}$$

When the density is constant, we call the center of mass the centroid of the object. To find the centroid we set $\delta = 1$