MA140-Engineering Calculus

Lecture 29

November 8, 2017

Example 1.1

The cable of a suspension bridge takes the shape of the curve:

$$y = \frac{h}{l^2}x^2 - \frac{2h}{l}x + h$$

Where $0 \leq x \leq 2l$, $h > 0$. Find the length of the cable.

$$\frac{dy}{dx} = \frac{2h}{l^2}x - \frac{2h}{l} = \frac{2h}{l}(\frac{x}{l} - 1)$$

So:

$$(\frac{dy}{dx})^2 = \left[\frac{2h}{l}(\frac{x}{l} - 1)\right]^2$$

The length of the curve equals:

$$S = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}dx = \int_0^{2l} \sqrt{1 + \left[\frac{2h}{l}(\frac{x}{l} - 1)\right]^2}dx$$

First we find the following indefinite integral:

$$\int \sqrt{1 + \left[\frac{2h}{l}(\frac{x}{l} - 1)\right]^2}dx, \quad (\star)$$

Let $u = \frac{2h}{l}(\frac{x}{l} - 1)$, then $du = \frac{2h}{l^2}dx$ or $\frac{l^2}{2h}du = dx$
So:

$$(\star) = \frac{l^2}{2h}\int\sqrt{1+u^2}du$$

quick reminder:

- $\sinh(x) = \frac{e^x - e^{-x}}{2}$
- $\cosh(x) = \frac{e^x + e^{-x}}{2}$
- $\frac{d}{dx}\sinh(x) = \cosh(x)$
- $\frac{d}{dx}\cosh(x) = \sinh(x)$
- $\cosh^2(x) - \sinh^2(x) = 1$

Now let $u = \sinh(v)$, so $du = \cosh(v)dv$, so:

$$(\star) = \frac{l^2}{2h}\int \sqrt{1+u^2}du = \frac{l^2}{2h}\int \sqrt{1+\sinh^2(v)}\cosh(v)dv$$

$$= \frac{l^2}{2h}\int \cosh(v)\cosh(v)dv = \frac{l^2}{2h}\int \cosh^2(v)dv$$

We have:

$$\cosh^2(v) = \frac{[e^{2v}+e^{-2v}+2]}{4} = \frac{1}{2}[\frac{e^{2v}+e^{-2v}+2}{2}]$$

$$= \frac{1}{2}[\frac{e^{2v}+e^{-2v}}{2}+1] = \frac{1}{2}[\cosh(2v)+1]$$

So the above integral is:

$$(\star) = \frac{l^2}{2h}\int \frac{1}{2}[\cosh(2v)+1]dv = \frac{l^2}{4h}\left[\frac{1}{2}\sinh(2v)+v\right]+c$$

So:

$$(\star) = \frac{l^2}{4h}\Big[\sinh(v)\sqrt{1+\sinh^2(v)}+v\Big]+c = \frac{l^2}{4h}\Big[u\sqrt{1+u^2}+\sinh^{-1}(u)\Big]+c$$

$$= \frac{l^2}{4h}\Big[\frac{2h}{l}(\frac{x}{l}-1)\sqrt{1+(\frac{2h}{l}(\frac{x}{l}-1))^2}+\sinh^{-1}(\frac{2h}{l}(\frac{x}{l}-1))\Big]+c$$

So the length of the cable is:

$$\frac{l^2}{4h}\Big[\frac{2h}{l}(\frac{x}{l}-1)\sqrt{1+(\frac{2h}{l}(\frac{x}{l}-1))^2}+\sinh^{-1}(\frac{2h}{l}(\frac{x}{l}-1))\Big]+c\Big]_0^{2l}$$

$$= \Big(\frac{l^2}{4h}(\frac{2h}{l}\sqrt{1+(\frac{2h}{l})^2}+\sinh^{-1}(\frac{2h}{l})\Big)$$

$$-\Big(\frac{l^2}{4h}(\frac{-2h}{l}\sqrt{1+(\frac{2h}{l})^2}+\sinh^{-1}(\frac{-2h}{l})\Big)$$

Using the fact that $\sinh^{-1}(-x) = -\sinh^{-1}(x)$, we can simplify the above answer:

$$= \sqrt{l^2 + 4h^2} + \frac{l^2}{2h}\sinh^{-1}(\frac{2h}{l})$$