MA140-Engineering Calculus

Lecture 23

October 25, 2017

Example 1.1

Evaluate

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$$I = \frac{2}{3} \int \frac{1}{1 + u^2} (\frac{\sqrt{3}}{2} du)$$

$$I = \frac{2}{3}\frac{\sqrt{3}}{2}\int \frac{1}{1+u^2}du$$

So

$$I = \frac{1}{\sqrt{3}}\tan^{-1}u + c = \frac{1}{\sqrt{3}}\tan^{-1}(\frac{2}{\sqrt{3}}x) + c$$

To compute the area of the region bounded by the graph of a function $y = f(x)$ and the $x-$axis requires more care when the function takes on both positive and negative values. We must be careful to break up the interval $[a, b]$ into subintervals on which the function doesnt change sign. Otherwise we might get cancellation between positive and negative signed areas, leading to an incorrect total.

Definition 1.2

To find the area between the graph of $y = f(x)$ and the $x-$axis over the interval $[a, b]$, do the following:

(1) Subdivide $[a, b]$ at the zeros of f

(2) Integrate f over each subinterval.

(3) Add the absolute values of the integrals.

Example 1.3

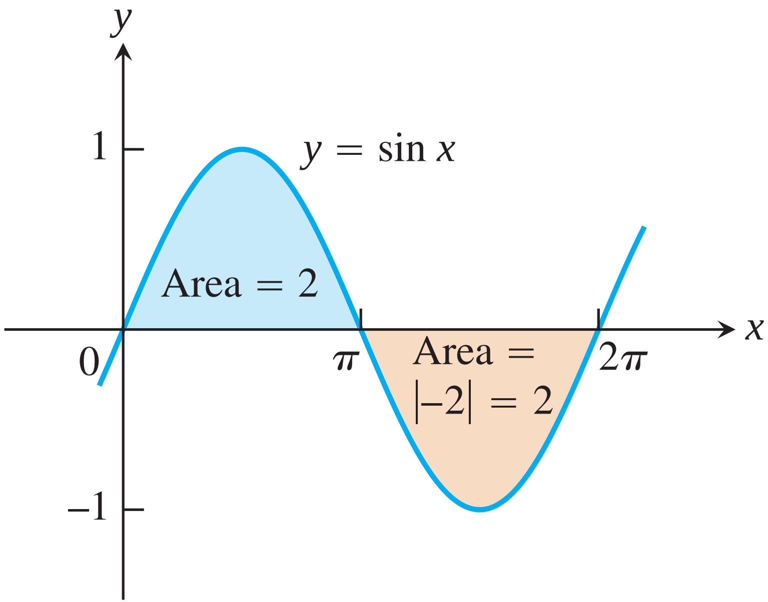
Suppose that $f(x) = \sin(x)$, find:

- $$\int_0^{2\pi} \sin(x)dx$$
- The area between the graph of $f(x)$ and the $x-$axis over $[0, 2\pi]$

The definite integral for $f(x) = \sin(x)$ is given by

$$\int_0^{2\pi} \sin(x)dx = -\cos(x)\Big]_0^{2\pi} = -[\cos(2\pi) - \cos(0)] = -[1 - 1] = 0$$

The definite integral is zero because the portions of the graph above and below the x-axis make canceling contributions.


y
1
–1
0
π
2π
x
y = sin x
Area = 2
Area =
|–2| = 2

The area between the graph of $f(x)$ and the $x-$axis over $[0, 2\pi]$ is calculated by breaking up the domain of $\sin(x)$ into two pieces: the interval $[0, \pi]$ over which it is nonnegative and the interval $[\pi, 2\pi]$ over which it is nonpositive.

$$\int_0^{\pi} \sin(x)dx = -\cos(x)\Big]_0^{\pi} = -[\cos(\pi) - \cos(0)] = -[-1 - 1] = 2$$

$$\int_{\pi}^{2\pi} \sin(x)dx = -\cos(x)\Big]_{\pi}^{2\pi} = -[\cos(2\pi) - \cos(\pi)] = -[1 - (-1)] = -2$$

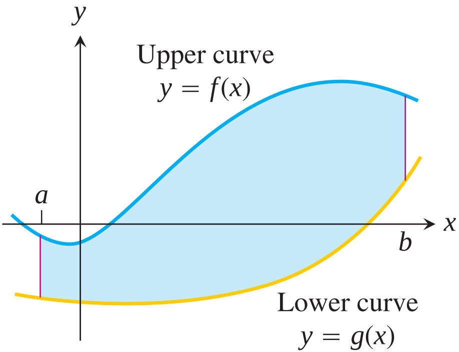
The second integral gives a negative value. The area between the graph and the axis is obtained by adding the absolute values

$$area = |2| + |-2| = 4$$

Definition 1.4

If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]dx$$



Example 1.5

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$

First we sketch the two curves

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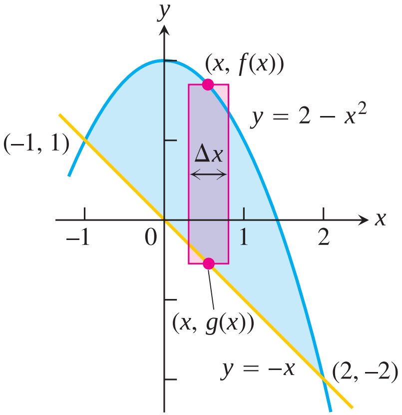

The limits of integration are found by solving $y = 2 - x^2$ and $y = -x$ simultaneously for x.

$$2 - x^2 = -x \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x+1)(x-2) = 0 \Rightarrow x = -1, x = 2$$

The region runs from $x = -1$ to $x = 2$. The limits of integration are $a = -1$, $b = 2$. The area between the curve is

$$A = \int_a^b [f(x) - g(x)]dx = \int_{-1}^{2} [(2 - x^2) - (-x)]dx$$

$$= \int_{-1}^{2} (2 + x - x^2)dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2}$$

Example 1.6

Find the area enclosed between the two curves $f(x) = 6 - 2x^2$ and $g(x) = 4x$.