MA140-Engineering Calculus

Lecture 35

November 27, 2019

First Order Linear Differential Equations (The Integrating Factor Method):

Definition 1.1

A D.E. of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

is called a **first order linear D.E.**, where $P(x)$ and $Q(x)$ are functions of x.

We can find the general solution of this D.E. as follows:

1.) Find the **integrating factor**

$$e^{\int P(x)dx}$$

2.) Multiply the D.E. by the integrating factor (I.F.) to get:

$$e^{\int P(x)dx}\left(\frac{dy}{dx} + P(x)y\right) = e^{\int P(x)dx}Q(x) \quad (\star)$$

3.) Note that the L.H.S. of ($\star$) equals:

$$\frac{d}{dx}\left(ye^{\int P(x)dx}\right)$$

So ($\star$) becomes:

$$d\left(ye^{\int P(x)dx}\right) = e^{\int P(x)dx}Q(x)dx$$

4.) Integrate both sides and solve for y.
This algorithm is called the **integrating factor method**.

Example 1.2

Solve the first order linear differential equation

$$\frac{dy}{dx} - 3y = 0$$

using the integrating factor method.

1.)

$$I.F. = e^{\int P(x)dx} = e^{\int -3dx} = e^{-3x}$$

(note that we do not include the arbitrary constant C).
2.) Multiply the D.E. by the I.F. to get

$$e^{-3x}(\frac{dy}{dx} - 3y) = 0e^{-3x} \quad (\star)$$

3.) Recall that the L.H.S. of $(\star)$ equals

$$\frac{d}{dx}(ye^{-3x})$$

therefore

$$\frac{d}{dx}(ye^{-3x}) = 0 \Rightarrow d(ye^{-3x}) = 0dx$$

4.) Integrate and solve for y:

$$\int d(ye^{-3x}) = \int 0dx \Rightarrow ye^{-3x} = 0 + C \Rightarrow y = Ce^{3x}$$

Example 1.3

Solve the first order linear differential equation

$$\frac{dy}{dx} - y = e^x$$

using the integrating factor method.

1.)

$$I.F. = e^{\int P(x)dx} = e^{\int -1dx} = e^{-x}$$

(note that we do not include the arbitrary constant C).

2.) Multiply the D.E. by the I.F. to get

$$e^{-x}(\frac{dy}{dx} - y) = e^x \cdot e^{-x} \quad (\star)$$

3.) Recall that the L.H.S. of $(\star)$ equals

$$\frac{d}{dx}(ye^{-x})$$

therefore

$$\frac{d}{dx}(ye^{-x}) = 1 \Rightarrow d(ye^{-x}) = 1dx$$

4.) Integrate and solve for y:

$$\int d(ye^{-x}) = \int 1dx \Rightarrow ye^{-x} = x + C \Rightarrow y = xe^{x} + c$$

Example 1.4

Solve the first order linear differential equation

$$\frac{dy}{dx} + 2xy = x$$

using the integrating factor method.

1. Note that $P(x) = 2x$ while $Q(x) = x$.
2. $$I.F. = e^{\int P(x)dx} = e^{\int 2xdx} = e^{x^2}$$
   (note that we do not include the arbitrary constant C).
3. Multiply the D.E. by the I.F. to get
   $$e^{x^2}(\frac{dy}{dx} + 2xy) = e^{x^2}x = \frac{d}{dx}\left(ye^{x^2}\right)$$

$$\implies ye^{x^2} = \int e^{x^2} x dx = e^{x^2}/2 + C$$

$$\implies y = \frac{e^{x^2}/2 + C}{e^{x^2}} = \frac{e^{x^2} + C}{2e^{x^2}}$$

Example 1.5

Solve the first order linear differential equation

$$\frac{dy}{dx} + \frac{2y}{x} = \frac{\sin x}{x^2}$$

$P(x) = 2/x$; $Q(x) = (\sin x)/x^2$, so we have

$$\int P(x)dx = \int \left(\frac{2}{x}\right) dx = 2\int \left(\frac{1}{x}\right) dx = 2\log x = \log(x^2)$$

Integrating Factor $= e^{\int P(x)dx} = e^{\log(x^2)} = x^2$

$$\implies x^2 \left[\frac{dy}{dx} + \frac{2y}{x}\right] = x^2 \left[\frac{\sin x}{x^2}\right]$$

$$\implies \frac{d}{dx}\left(yx^2\right) = \sin x$$

$$\implies yx^2 = \int \sin x dx = -\cos x + C$$

$$\implies y = \frac{-\cos x + C}{x^2}$$

Last year exam Solutions:

Question $1.(a).i$:

$$\lim_{x\to 1} \frac{x-1}{(\sqrt{x}-1)(x+2)} = \lim_{x\to 1} \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(x+2)}$$

$$= \lim_{x\to 1} \frac{\sqrt{x}+1}{x+2} = \frac{2}{3}$$

Question $1.(a).ii$:

we apply L'Hôpital's Rule:

$$\lim_{\theta\to 0} \frac{6\sin\theta}{\theta + 2\tan\theta} \overset{\text{H}}{=} \lim_{\theta\to 0} \frac{6\cos\theta}{1 + 2\sec^2\theta} = \frac{6}{3} = 2$$

Question $2.(a).i$:

The derivative of the function $f(x)$ with respect to the variable x is the function f' or $\frac{df}{dx}$ whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Question $2.(a).ii$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 + (x+h) - (x^2 + x)}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + h^2 + 2xh + x + h - x^2 - x}{h} = \lim_{h \to 0} \frac{h^2 + 2xh + h}{h}$$

$$= \lim_{h \to 0} (h + 2x + 1) = 2x + 1$$

Question $2.(b).i$:

$$f(x) = e^{\cos x^2} \sin x$$

$$\Rightarrow f'(x) = (e^{\cos x^2} \sin x)' = (e^{\cos x^2})' \sin x + e^{\cos x^2} (\sin x)' \quad (\star)$$

To differentiate $e^{\cos x^2}$ we use the chain rule:
Let $y = e^{\cos x^2} = e^u$, where $u = \cos x^2$, then by chain rule we have:

$$y' = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot (\cos x^2)'$$

To differentiate $\cos x^2$ we use the chain rule again:
$u = \cos x^2 = \cos v$, where $v = x^2$, then by chain rule we have:

$$u' = \frac{du}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx} = (-\sin v)(2x) = (-\sin x^2)(2x)$$

So

$$(e^{\cos x^2})' = y' = e^u \cdot (-\sin x^2)(2x) = (e^{\cos x^2}) \cdot (-\sin x^2)(2x)$$

Therefore

$$(\star) = f'(x) = (e^{\cos x^2}) \cdot (-\sin x^2)(2x)\sin x + e^{\cos x^2}(\cos x)$$

Question $2.(b).iii$:

$$y = (\cos x)^x$$

$$\Rightarrow \ln y = \ln(\cos x)^x \Rightarrow \ln y = x \ln \cos x$$

$$\Rightarrow (\ln y)' = (x \ln \cos x)' \Rightarrow \frac{y'}{y} = 1 \cdot \ln \cos x + x \cdot \frac{(\cos x)'}{\cos x}$$

$$\Rightarrow \frac{y'}{y} = \ln \cos x - \frac{x \sin x}{\cos x}$$

$$\Rightarrow y' = y\left[\ln \cos x - \frac{x \sin x}{\cos x}\right] = (\cos x)^x\left[\ln \cos x - \frac{x \sin x}{\cos x}\right]$$