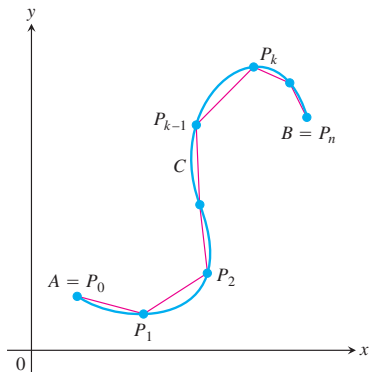


# MA140-Engineering Calculus

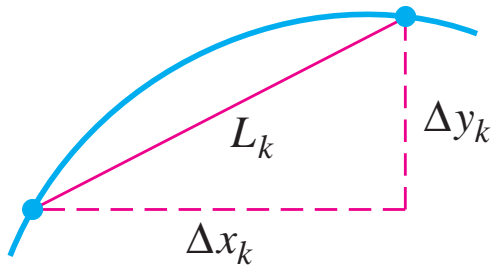
## Lecture 28

November 12, 2019

**Arc Length:** Let  $C$  be a curve given by the equation  $y = f(x)$ . It may be helpful to imagine the curve as the path of a particle moving from point  $A$  to point  $B$ . We subdivide the path (or arc)  $AB$  into  $n$  pieces at points  $A = P_0, P_1, P_2, \dots, P_n = B$ . Join successive points of this subdivision by straight line segments



A representative line segment:



has length:

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

An intuitive approximation to the length of the curve  $AB$ ,  $S$ , is the sum of all the lengths  $L_k$  :

$$S \cong \sum_{k=1}^n L_k = \sum_{k=1}^n \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

So

$$S \cong \sum_{k=1}^n \sqrt{(\Delta x_k)^2 \left(1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2\right)}$$

So:

$$S \cong \sum_{k=1}^n \sqrt{\left(1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2\right)} (\Delta x_k)$$

Using a Riemann sum approach.

Let  $\Delta x \rightarrow 0$  or  $n \rightarrow \infty$ , we get:

$$S = \lim_{n \rightarrow \infty} \sum_{k=1}^n \sqrt{\left(1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2\right)} (\Delta x_k)$$

So:

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

## Theorem 1.1

If  $f$  is continuously differentiable on the closed interval  $[a, b]$ , the length of the curve (graph)  $y = f(x)$  from  $x = a$  to  $x = b$  is:

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

## Example 1.2

Find the length of the curve

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \quad 0 \leq x \leq 1$$

We use the theorem with  $a = 0$ ,  $b = 1$ , and

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1$$

First we take the derivative of  $y$ :

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2}x^{1/2} = 2\sqrt{2}x^{1/2}$$

So

$$\left(\frac{dy}{dx}\right)^2 = (2\sqrt{2}x^{1/2})^2 = 8x$$

The length of the curve from  $x = 0$  to  $x = 1$  is:

$$\begin{aligned} S &= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^1 \sqrt{1 + 8x} dx \\ &= \frac{2}{3} \cdot \frac{1}{8} (1 + 8x)^{3/2} \Big|_0^1 = \frac{13}{6} \end{aligned}$$

### Example 1.3

The cable of a suspension bridge takes the shape of the curve:

$$y = \frac{h}{l^2}x^2 - \frac{2h}{l}x + h$$

Where  $0 \leq x \leq 2l$ ,  $h > 0$ . Find the length of the cable.

$$\frac{dy}{dx} = \frac{2h}{l^2}x - \frac{2h}{l} = \frac{2h}{l}\left(\frac{x}{l} - 1\right)$$

So:

$$\left(\frac{dy}{dx}\right)^2 = \left[\frac{2h}{l}\left(\frac{x}{l} - 1\right)\right]^2$$

The length of the curve equals:

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^{2l} \sqrt{1 + \left[\frac{2h}{l}\left(\frac{x}{l} - 1\right)\right]^2} dx$$

First we find the following indefinite integral:

$$\int \sqrt{1 + \left[\frac{2h}{l}\left(\frac{x}{l} - 1\right)\right]^2} dx, \quad (\star)$$

Let  $u = \frac{2h}{l}(\frac{x}{l} - 1)$ , then  $du = \frac{2h}{l^2}dx$  or  $\frac{l^2}{2h}du = dx$   
So:

$$(\star) = \frac{l^2}{2h} \int \sqrt{1 + u^2} du$$

quick reminder:

- $\sinh(x) = \frac{e^x - e^{-x}}{2}$
- $\cosh(x) = \frac{e^x + e^{-x}}{2}$
- $\frac{d}{dx} \sinh(x) = \cosh(x)$
- $\frac{d}{dx} \cosh(x) = \sinh(x)$
- $\cosh^2(x) - \sinh^2(x) = 1$