Partial Differentiation II

Last time we saw

- Functions and derivatives in 1D & 2D
- Definitions of derivatives
- Examples

Today we will discuss

- Notation
- Partial differentiation
- Mixed derivatives
- Taylor expansion

Partial derivatives

Notation:

$$f'(x) = f_x(x) = \tfrac{df}{dx}(x)$$

$$f'_x(x,y) = f_x(x,y) = \tfrac{\partial f}{\partial x}(x,y)$$

$$f'_y(x,y) = f_y(x,y) = \tfrac{\partial f}{\partial y}(x,y).$$

Example 5: Let $f(x,y) = x^2y^3$ then

$$f_x(x,y) = 2xy^3 \text{ and } f_y(x,y) = 3x^2y^2.$$

At the point $(x,y) = (1,2)$ we have

$$f(1,2) = 8, \quad \frac{\partial f}{\partial x}(1,2) = 16, \quad \frac{\partial f}{\partial y}(1,2) = 12.$$

Partial derivatives

Notation:

$$f''_{xx}(x,y) = f_{xx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}(x,y)\right) = \frac{\partial^2 f}{\partial x^2}(x,y)$$
$$f''_{yy}(x,y) = f_{yy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}(x,y)\right) = \frac{\partial^2 f}{\partial y^2}(x,y)$$
$$f''_{xy}(x,y) = f_{xy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right) = \frac{\partial^2 f}{\partial y \partial x}(x,y)$$
$$f''_{yx}(x,y) = f_{yx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right) = \frac{\partial^2 f}{\partial x \partial y}(x,y)$$

Example 5 continued: The second derivatives of $f(x,y) = x^2y^3$ are

$$f_{xx}(x,y) = 2y^3,\ f_{yy}(x,y) = 6x^2y,\ f_{xy}(x,y) = 6xy^2,\ f_{yx}(x,y) = 6xy^2.$$

Notice that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ or $f_{yx} = f_{xy}$.

Three variables

Example 6: Let $f(x,y,z) = x^2yz - xyz^2$, then

$$\frac{\partial f}{\partial x} = 2xyz - yz^2, \;\; \frac{\partial f}{\partial y} = x^2z - xz^2, \;\; \frac{\partial f}{\partial z} = x^2y - 2xyz,$$

$$\frac{\partial^2 f}{\partial x^2} = 2yz, \;\; \frac{\partial^2 f}{\partial y^2} = 0, \;\; \frac{\partial^2 f}{\partial z^2} = -2xy,$$

$$\begin{aligned} \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial^2 f}{\partial x \partial y} = 2xz - z^2, \\ \frac{\partial^2 f}{\partial z \partial x} &= \frac{\partial^2 f}{\partial x \partial z} = 2xy - 2yz, \\ \frac{\partial^2 f}{\partial y \partial z} &= \frac{\partial^2 f}{\partial z \partial y} = x^2 - 2xz, \end{aligned}$$

$$\frac{\partial^3 f}{\partial z \partial y \partial x} = \frac{\partial^3 f}{\partial y \partial x \partial z} = \frac{\partial^3 f}{\partial x \partial z \partial y} = 2x - 2z.$$

Partial derivatives

Theorem 1: *If $f = f(x,y)$ is a sufficiently smooth function then*

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

Proof: From the definition we have

$$\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \lim_{\Delta x \to 0} \frac{\frac{\partial f}{\partial y}(x+\Delta x, y) - \frac{\partial f}{\partial y}(x,y)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\lim_{\Delta y \to 0} \frac{f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y)}{\Delta y} - \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}\right) \\
&= \lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \frac{f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y) - f(x, y+\Delta y) + f(x,y)}{\Delta x \Delta y} \\
&= \lim_{\Delta y \to 0} \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y) - f(x, y+\Delta y) + f(x,y)}{\Delta x \Delta y} \\
&= \lim_{\Delta y \to 0} \frac{1}{\Delta y}\left(\lim_{\Delta x \to 0} \frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x} - \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}\right) \\
&= \lim_{\Delta y \to 0} \frac{\frac{\partial f}{\partial x}(x, y+\Delta y) - \frac{\partial f}{\partial x}(x,y)}{\Delta y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}. \qquad \blacksquare
\end{aligned}$$

Partial derivatives

Another way to show that the mixed second derivatives are equal is to have a geometrical view point. We have that

$$f(x_0+\Delta x, y_0) \approx f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\Delta x, \tag{1}$$

$$f(x_0+\Delta x, y_0+\Delta y) \approx f(x_0+\Delta x,y_0) + \frac{\partial f}{\partial y}(x_0+\Delta x,y_0)\Delta y. \tag{2}$$

From (1) and (2) we get

$$\begin{aligned}
\frac{\partial f}{\partial y}(x_0+\Delta x, y_0) &\approx \frac{\partial f}{\partial y}(x_0,y_0) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x_0,y_0)\right)\Delta x \\
f(x_0+\Delta x, y_0+\Delta y) &\approx f(x_0+\Delta x,y_0) + \left(\frac{\partial f}{\partial y}(x_0,y_0) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x_0,y_0)\right)\Delta x\right)\Delta y \\
&\approx f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\Delta x + \frac{\partial f}{\partial y}(x_0,y_0)\Delta y + \frac{\partial^2 f}{\partial y \partial x}(x_0,y_0)\Delta x \Delta y
\end{aligned}$$

The derivatives have been done in the following order:

$$(x_0,y_0) \longrightarrow (x_0+\Delta x, y_0) \longrightarrow (x_0+\Delta x, y_0+\Delta y)$$

Partial differentiation

In the same way we have

$$(x_0, y_0) \longrightarrow (x_0, y_0 + \Delta y) \longrightarrow (x_0 + \Delta x, y_0 + \Delta y)$$

which gives

$$f(x_0+\Delta x, y_0+\Delta y) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)\Delta x + \frac{\partial f}{\partial y}(x_0, y_0)\Delta y + \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)\Delta y \Delta x$$

and so

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

Taylor expansions

Taylor's theorem in **one** dimension about $x = x_0$:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n-1)}(x_0)}{(n-1)!}(x - x_0)^{n-1} + R_n(x - x_0)$$

Taylor expansion in **two** dimensions:

First we have

$$f(x_0 + \Delta x, y_0) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)\Delta x + \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}(x_0, y_0)\Delta x^2,$$

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0 + \Delta x, y_0) + \frac{\partial f}{\partial y}(x_0 + \Delta x, y_0)\Delta y + \frac{1}{2!}\frac{\partial^2 f}{\partial y^2}(x_0 + \Delta x, y_0)\Delta y^2,$$

$$\frac{\partial f}{\partial y}(x_0 + \Delta x, y_0) \approx \frac{\partial f}{\partial y}(x_0, y_0) + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x_0, y_0)\right)\Delta x,$$

which gives

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)\Delta x + \frac{\partial f}{\partial y}(x_0, y_0)\Delta y$$
$$+ \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}(x_0, y_0)\Delta x^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)\Delta x \Delta y + \frac{\partial^2 f}{\partial y^2}(x_0, y_0)\Delta y^2\right)$$

Taylor expansions

The Taylor expansion about $(x, y) = (x_0, y_0)$ can now be written as

$$\begin{aligned} f(x,y) =& f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\Delta x + \frac{\partial f}{\partial y}(x_0,y_0)\Delta y \\ &+ \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}(x_0,y_0)\Delta x^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)\Delta x \Delta y + \frac{\partial^2 f}{\partial y^2}(x_0,y_0)\Delta y^2\right) \\ &+ \cdots + \frac{1}{(n-1)!}\left(\ldots\right) + R_n(x_0,y_0) \\ =& \sum_{j=0}^{n-1} \frac{1}{j!}\left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right)^j f(x_0,y_0) + R_n(x_0,y_0), \end{aligned}$$

where $\Delta x = x - x_0$ and $\Delta y = y - y_0$.