

Mathematical Methods 1 MP231

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Topics

- Partial Differentiation – 5 lectures
- Stationary Points – 3 lectures
- Optimisation & Lagrange Multipliers – 2 lectures
- Fourier Series – 4 lectures
- Multiple Integration & Green's Theorem – 7 Lectures

Tutorials

Problem sheets will be handed out (about) every second Monday and the tutor (Neil Humphries) will go through a selection of problems.

Partial Differentiation I

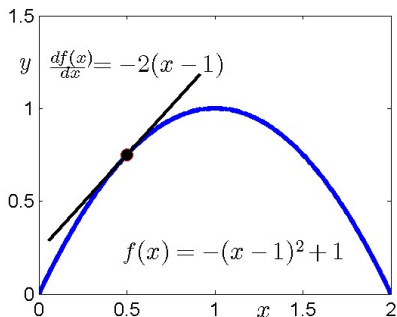
Today we will discuss

- Functions and derivatives in 1D & 2D
- Definitions of derivatives
- Examples

Functions and derivatives - 1D

Consider the function $y = f(x)$, where $f(x) = -(x - 1)^2 + 1$. Then we have that the slope at any given point x is given by

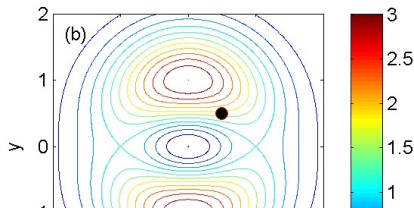
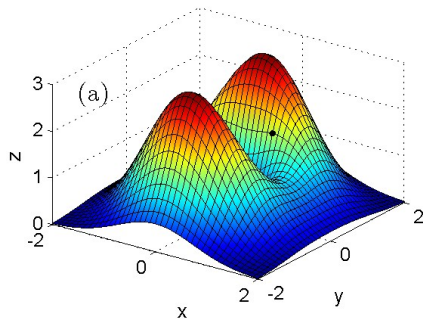
$$f'(x) = \frac{df}{dx}(x) = -2(x - 1).$$



At $x = 0.5$ we have that $f'(0.5) = 1 > 0$. What does this mean?

Functions and derivatives - 2D

Consider the function $z = f(x, y)$, where $f(x, y) = (x^2 + 3y^2)e^{1-x^2-y^2}$.



Functions and derivatives - 2D

For $y = 0.5$ we can write

$$g(x) = f(x, 0.5) = (0.75 + x^2)e^{0.75-x^2}$$

and for $x = 0.5$ we can write

$$h(y) = f(0.5, y) = (0.25 + 3y^2)e^{0.75-y^2}.$$

Thus

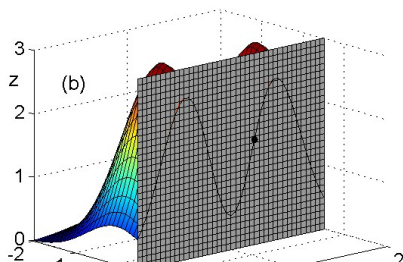
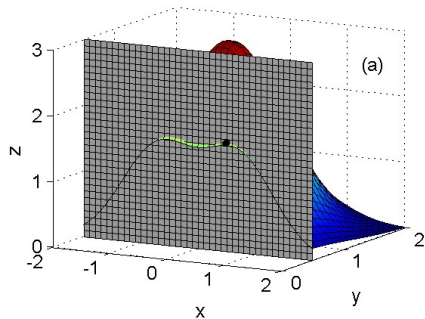
$$g'(x) = f'_x(x, 0.5) = (0.5x - 2x^3)e^{0.75-x^2} = 0.5x(1 - 4x^2)e^{0.75-x^2}$$

and

$$h'(y) = f'_y(0.5, y) = (5.5y - 6y^3)e^{0.75-y^2} = y(5.5 - 6y^2)e^{0.75-y^2},$$

which yield $g'(0.5) = 0$ and $h'(0.5) \approx 3.30$. This is depicted on the next slide.

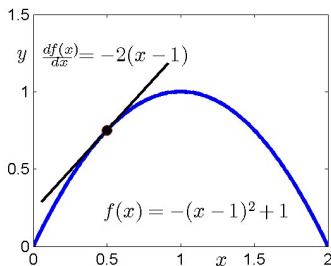
Functions and derivatives - 2D



Functions of **one** variable

For the function $f(x) = -(x - 1)^2 - 1$ we have that

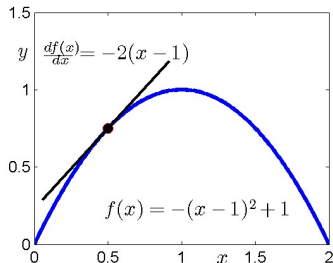
$$\frac{df(x)}{dx} = -2(x - 1) = 2(1 - x) = 2 - 2x$$



Definition: The derivative of a function $f(x)$.

$$\frac{df(x)}{dx} \stackrel{\text{def}}{=} \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Functions of **one** variable



$$\begin{aligned}\frac{df(x)}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-(x + \Delta x - 1)^2 + 1 + (x - 1)^2 - 1}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{-(x + \Delta x - 1)^2 + (x - 1)^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-(x^2 + (\Delta x)^2 + 2x\Delta x - 2x - 2\Delta x) + (x^2 - 2x + 1)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-(\Delta x)^2 - 2x\Delta x + 2\Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0} -\Delta x - 2x + 2 = -2(x - 1)\end{aligned}$$

Functions of **two** variables

Definition: The partial derivatives of a function $f(x, y)$.

$$\frac{\partial f(x, y)}{\partial x} \stackrel{\text{def}}{=} \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$\frac{\partial f(x, y)}{\partial y} \stackrel{\text{def}}{=} \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

Example 1: Let $f(x, y) = x^2 y^3$ then

$$\begin{aligned} \frac{\partial f(x, y)}{\partial x} &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 y^3 - x^2 y^3}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x^2 + 2x\Delta x + (\Delta x)^2) y^3 - x^2 y^3}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(2x\Delta x + (\Delta x)^2) y^3}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (2x + \Delta x) y^3 = 2xy^3 \end{aligned}$$

Functions of **two** variables

Example 1 continued:

$$\begin{aligned}\frac{\partial f(x, y)}{\partial y} &= \lim_{\Delta y \rightarrow 0} \frac{x^2(y + \Delta y)^3 - x^2y^3}{\Delta y} \\ &= \lim_{\Delta y \rightarrow 0} \frac{x^2(y^3 + 3y^2\Delta y + 3y(\Delta y)^2 + \Delta y^3) - x^2y^3}{\Delta y} \\ &= \lim_{\Delta y \rightarrow 0} \frac{x^2(3y^2\Delta y + 3y(\Delta y)^2 + \Delta y^3)}{\Delta y} \\ &= \lim_{\Delta y \rightarrow 0} x^2(3y^2 + 3y\Delta y + (\Delta y)^2) = 3x^2y^2\end{aligned}$$

Notice that

$$\frac{\partial f(x, y)}{\partial x} \neq \frac{\partial f(x, y)}{\partial y}!$$

Functions of **two** variables

Example 2: Let $f(x, y) = x^3 + y^3 + 3x^4y^5 + 100$

$$\begin{aligned}\frac{\partial f(x, y)}{\partial x} &= \frac{\partial}{\partial x} (x^3 + y^3 + 3x^4y^5 + 100) \\ &= 3x^2 + 0 + 3(4x^3)y^5 + 0 = 3x^2 + 12x^3y^5\end{aligned}$$

$$\frac{\partial}{\partial y} \left(\frac{\partial f(x, y)}{\partial x} \right) = \frac{\partial}{\partial y} (3x^2 + 12x^3y^5) = 0 + 12x^3(5y^4) = \underline{60x^3y^4}$$

$$\begin{aligned}\frac{\partial f(x, y)}{\partial y} &= \frac{\partial}{\partial y} (x^3 + y^3 + 3x^4y^5 + 100) \\ &= 0 + 3y^2 + 3x^4(5y^4) + 0 = 3y^2 + 15x^4y^4\end{aligned}$$

$$\frac{\partial}{\partial x} \left(\frac{\partial f(x, y)}{\partial y} \right) = \frac{\partial}{\partial x} (3y^2 + 15x^4y^4) = 0 + 15(4x^3)y^4 = \underline{60x^3y^4}$$

Notice that

$$\frac{\partial}{\partial y} \left(\frac{\partial f(x, y)}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial f(x, y)}{\partial y} \right)!$$

Functions of **two** variables

Example 3: Let $f(x, y) = Ax^4 + Bx^2y^2 + Cy^4$, where A, B and C are arbitrary constants, and show that

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 4f. \quad (*)$$

Soln: First we calculate

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x} (Ax^4 + Bx^2y^2 + Cy^4) \\ &= A(4x^3) + B(2x)y^2 + 0 = 4Ax^3 + 2Bxy^2, \\ \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y} (Ax^4 + Bx^2y^2 + Cy^4) \\ &= 0 + Bx^2(2y) + C(4y^3) = 2Bx^2y + 4Cy^3, \end{aligned}$$

and then the left-hand side of (*).

Functions of **two** variables

Example 3 continued:

Thus

$$\begin{aligned}x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} &= x(4Ax^3 + 2Bxy^2) + y(2Bx^2y + 4Cy^3) \\&= 4Ax^4 + 2Bx^2y^2 + 2Bx^2y^2 + 4Cy^4 \\&= 4(Ax^4 + Bx^2y^2 + Cy^4) \\&= 4f\end{aligned}$$

and so

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 4f.$$

Note: This is an example of a **partial differential equation** (PDE).

Functions of **two** variables

Example 4: Let $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$ and show that the **Cauchy-Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are satisfied.

Soln: First we calculate

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} (x^2 - y^2) = 2x,$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y} (2xy) = 2x,$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} (x^2 - y^2) = -2y,$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x} (2xy) = 2y.$$

Now it is clear that the Cauchy-Riemann equations are satisfied.

Functions of **two** variables

Example 4 continued: A consequence of the Cauchy-Riemann equations is

$$\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial y} \right) = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial v}{\partial x} \right) = \frac{\partial}{\partial y} \left(-\frac{\partial u}{\partial y} \right)$$

and so

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \quad \Leftrightarrow \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

which is known as **Laplace's equation** (in two variables).

Notice that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right), \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right),$$
$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right) \quad \text{and} \quad \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} \right).$$