Lecture 5: Partial Fractions ctd.

MA140: Engineering Calculus.

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Partial Fractions

Recall from our last lecture:

Note: Any polynomial (with real coefficients) can be factorised fully into the product of

- linear
- and irreducible quadratic factors.

We get four different combinations of factors in the denominator. We looked at the first case, and how to find the partial fractions in this case. Let's continue with Case 2.

Partial Fractions

(2) Linear factors to the power greater than 1 in the denominator, (i.e repeated linear factors).

If $(x-\alpha)^k$ appears in the denominator, it will give rise to the following terms:

$$\frac{A_1}{x-\alpha}+\frac{A_2}{(x-\alpha)^2}+\ldots+\frac{A_k}{(x-\alpha)^k}$$

Exercise

Show that

$$\frac{3x+1}{(x-1)^2(x+2)}=\frac{\frac{5}{9}}{x-1}+\frac{\frac{4}{3}}{(x-1)^2}+\frac{-\frac{5}{9}}{x+2}$$

Partial Fractions

Solution

$$\frac{3x+1}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$$= \frac{A(x-1)(x+2) + B(x+2) + C(x-1)^2}{(x-1)^2(x+2)}$$

So $3x+1 = A(x-1)(x+2) + B(x+2) + C(x-1)^2$

$\underline{x=1} \Rightarrow 4 = B(3) \Rightarrow B = \frac{4}{3}$

$\underline{x=-2} \Rightarrow -5 = C(9) \Rightarrow C = -\frac{5}{9}$

Partial Fractions

Solution ctd.

So we have

$$3x+1 = A(x-1)(x+2) + \tfrac{4}{3}(x+2) - \tfrac{5}{9}(x-1)^2$$

Now we need to compare coefficients to find A.

$$3x+1 = A(x^2+x-2) + \tfrac{4}{3}x + \tfrac{8}{3} - \tfrac{5}{9}(x^2-2x+1)$$

$$= (A - \tfrac{5}{9})x^2 + (A + \tfrac{4}{3} + \tfrac{10}{9})x - 2A + \tfrac{8}{3} - \tfrac{5}{9}$$

$$\Rightarrow A - \tfrac{5}{9} = 0 \quad \Rightarrow A = \tfrac{5}{9} \checkmark$$

Check the other coefficients:

- $3 = A + \tfrac{4}{3} + \tfrac{10}{9} = \tfrac{5}{9} + \tfrac{4}{3} + \tfrac{10}{9} = \tfrac{27}{9}$ ✓
- $1 = -2A + \tfrac{19}{9} = -\tfrac{10}{9} + \tfrac{19}{9} = \tfrac{9}{9} = 1$ ✓

Partial Fractions

(3) Irreducible quadratic factors.

Irreducible quadratic factors can not be factorised using real numbers, e.g. $x^2 + x + 1$.

An irreducible quadratic factor $ax^2 + bx + c$ gives rise to partial fractions of the form

$$\frac{Ax + B}{ax^2 + bx + c}.$$

Example

$$\frac{5x}{(x^2 + x + 1)(x - 2)} = \frac{Ax + B}{x^2 + x + 1} + \frac{C}{x - 2}$$

Find A, B and C.

Partial Fractions

Example ctd.

$$\frac{5x}{(x^2+x+1)(x-2)} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x-2}$$

$$= \frac{(Ax+B)(x-2) + C(x^2+x+1)}{(x^2+x+1)(x-2)}$$

So, $5x = (Ax+B)(x-2) + C(x^2+x+1)$

Let $x=2$. Then

$$5(2) = 0 + C(2^2+2+1)$$

$$\Rightarrow 10 = C(7)$$

$$\Rightarrow C = 10/7$$

So we have

$$5x = (Ax+B)(x-2) + \frac{10}{7}(x^2+x+1)$$

or $5x = \left(A+\frac{10}{7}\right)x^2 + \left(B-2A+\frac{10}{7}\right)x + \left(-2B+\frac{10}{7}\right)$

Partial Fractions

Example ctd.

- Compare the constant terms on both sides:

$$0 = -2B + \frac{10}{7}$$

$$\Rightarrow 2B = \frac{10}{7} \qquad \Rightarrow B = \frac{5}{7}$$

- Coefficient of $x^2 = 0$, i.e.

$$0 = A + \frac{10}{7}$$

$$\Rightarrow A = -\frac{10}{7}$$

So we get

$$\frac{5x}{(x^2+x+1)(x-2)} = \frac{-\frac{10}{7}x + \frac{5}{7}}{x^2+x+1} + \frac{\frac{10}{7}}{x-2}$$

Check coefficient of x: $5 = B - 2A + \frac{10}{7}$?

$$\frac{5}{7} - 2\left(-\frac{10}{7}\right) + \frac{10}{7} = \frac{5}{7} + \frac{20}{7} + \frac{10}{7} = \frac{35}{7} = 5 \checkmark$$

(4) Irreducible quadratic factors to power greater than 1.

Each repeated irreducible quadratic factor $(ax^2 + bx + c)^k$ in the denominator will give rise to

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + ... + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}.$$