Lecture 4: Rational Functions / Partial Functions

MA140: Engineering Calculus.

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Exercise from last week

Sketch the graph of $y = x^3 - 6x^2 + 11x - 6$.

Solution

$x = 0 \Rightarrow y = -6 \to (0, -6)$ is y-intercept.

$y = 0 \Rightarrow 0 = x^3 - 6x^2 + 11x - 6$

$0 = (x-1)(\quad \ldots\ldots \quad)$

$0 = (x-1)(x^2 - 5x + 6)$

$0 = (x-1)(x-2)(x-3)$

$\Rightarrow x = 1, x = 2,$ or $x = 3$

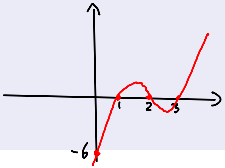
Find factors! (Try $x = 1$: $1^3 - 6(1^2) + 11(1) - 6 = 0$ ✓ so $(x-1)$ is factor)

Use Long Division:

$(x-1)\overline{)\,x^3 - 6x^2 + 11x - 6}$ gives $x^2 - 5x + 6$

Factorise $x^2 - 5x + 6$

So $(1,0), (2,0), (3,0)$ are x-intercepts.



Note: To revise how to solve cubic equation, check here:

http://www.mathtutor.ac.uk/algebra/cubicequations

Rational Functions

Rational Functions have the general form

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials.

- If degree of $p(x) <$ degree of $q(x)$, $f(x)$ is called a **strictly proper rational function**.
- If degree of $p(x) =$ degree of $q(x)$, $f(x)$ is called a **proper rational function**.
- If degree of $p(x) >$ degree of $q(x)$, $f(x)$ is called an **improper rational function**.

Rational Functions

It is often helpful to rewrite rational functions. There are several ways to do so.

An improper or proper rational function can always be expressed as a polynomial plus a strictly proper rational function, for example by algebraic division.

Example

$$\frac{3x^4 + 2x^3 - 5x^2 + 6x - 7}{x^2 - 2x + 3} = 3x^2 + 8x + 2 - \frac{14x + 13}{x^2 - 2x + 3}$$

Why? 🤔

Rational Functions

Example

Why? Use Long Division to divide numerator by denominator.

$$
\begin{array}{r|l}
 & 3x^2 + 8x + 2 \\
\hline
x^2 - 2x + 3 & 3x^4 + 2x^3 - 5x^2 + 6x - 7 \\
 & 3x^4 - 6x^3 + 9x^2 \\
\hline
 & 8x^3 - 14x^2 + 6x - 7 \\
 & 8x^3 - 16x^2 + 24x \\
\hline
 & 2x^2 - 18x - 7 \\
 & 2x^2 - 4x + 6 \\
\hline
 & -14x - 13
\end{array}
$$

So,

$$f(x) = 3x^2 + 8x + 2 - \frac{14x + 13}{x^2 - 2x + 3}$$

Partial Fractions

An algebraic fraction can often be written as a sum of simpler fractions. These are called **partial fractions.**

For example

$$\frac{8x - 12}{x^2 - 2x - 3}$$

can be written as

$$\frac{3}{x-3} + \frac{5}{x+1}$$

How do we find partial fractions?

First, factorise the denominator.

Partial Fractions

Note: Any polynomial (with real coefficients) can be factorised fully into the product of

- linear
- and irreducible quadratic factors.

We get different combinations of factors in the denominator. Let's look at **four cases**, and how to find the partial fractions in each case.

(1) Linear factors to the power of 1 in the denominator.

Example

$$\frac{3x}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} = \frac{A(x+2) + B(x-1)}{(x-1)(x+2)}$$

Partial Fractions

We have **two methods** to find A and B.

Method 1: Comparing coefficients

Example ctd.

$$\frac{3x}{(x-1)(x+2)} = \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$$

$$= \frac{Ax+2A+Bx-B}{(x-1)(x+2)}$$

So $$\frac{3x}{(x-1)(x+2)} = \frac{(A+B)x+2(A-B)}{(x-1)(x+2)}$$

$$\Rightarrow \left.\begin{array}{l} 3 = A+B \\ 0 = 2A-B \end{array}\right\} \text{and}$$

$$\Rightarrow \left.\begin{array}{l} 2A = B \\ 3 = A+2A \end{array}\right\} \text{and} \quad \Rightarrow \quad A=1 \text{ and } B=2$$

So $$\frac{3x}{(x-1)(x+2)} = \frac{1}{x-1} + \frac{2}{x+2}$$

Partial Fractions

Method 2: Substituting specific values for x.

Example ctd.

We are looking for A and B in

$$3x = A(x+2) + B(x-1)$$

Set $x=-2$. Then

$$3(-2) = A(0) + B(-3)$$

$$\Rightarrow -6 = -3B$$

$$\Rightarrow B = 2$$

Set $x = 1$. Then

$$3(1) = A(3) + B(0)$$

$$\Rightarrow 3 = 3A$$

$$\Rightarrow A = 1$$

✓

Partial Fractions

Note: Often a combination of the two methods is needed.

Example

Write

$$\frac{8x-12}{x^2-2x-3}$$

as sum of partial fractions.

Solution

- Factorise x^2-2x-3.

$$x^2-2x-3 = (x-3)(x+1)$$

- $$\frac{8x-12}{x^2-2x-3} = \frac{A}{x-3} + \frac{B}{x+1}$$

Partial Fractions

Solution ctd.

- $$\frac{8x-12}{x^2-2x-3} = \frac{A}{x-3} + \frac{B}{x+1}$$

$$= \frac{A(x+1) + B(x-3)}{(x-3)(x+1)}$$

$$= \frac{Ax + A + Bx - 3B}{(x-3)(x+1)}$$

$$= \frac{(A+B)x + (A-3B)}{(x-3)(x+1)}$$

- Compare coefficients. $8x - 12 = (A+B)x + (A-3B)$

Partial Fractions

Solution ctd.

so

$$\begin{cases} 8 = A + B \\ -12 = A - 3B \end{cases}$$

$$B = 8 - A$$

$$-12 = A - 3(8 - A)$$

$$= 4A - 24$$

$$A = 3$$

$$B = 5$$

Exercise

Find the constants A, B and C, so that

$$\frac{2x+1}{(x-2)(x+1)(x-3)} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{x-3}$$