Lecture 2: Introduction to Functions

MA140: Engineering Calculus.

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September 28, 2022

Functions

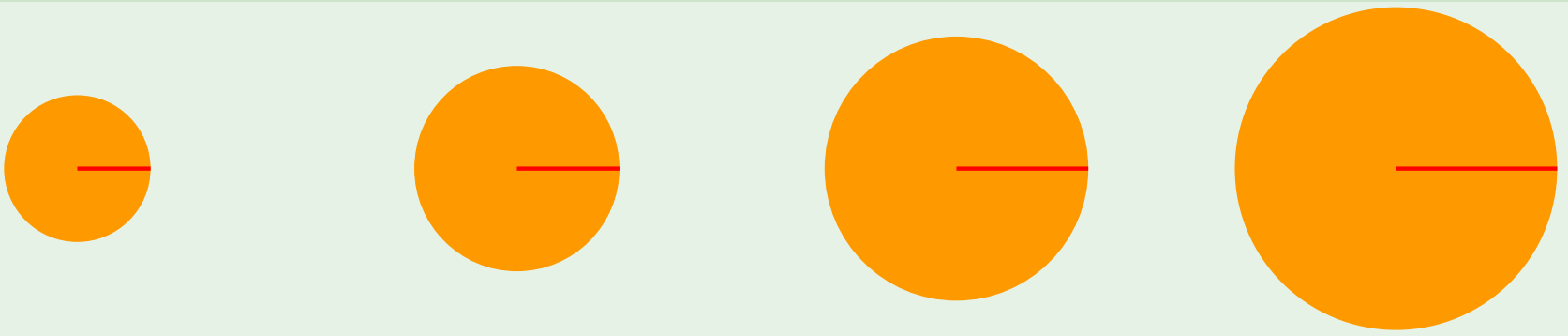
Question for you: What is a function in Mathematics?

Take a few minutes to answer the question.

Use your own words – and don't look it up on Google ;-)

Functions

Functions arise whenever one quantity **depends** on another.

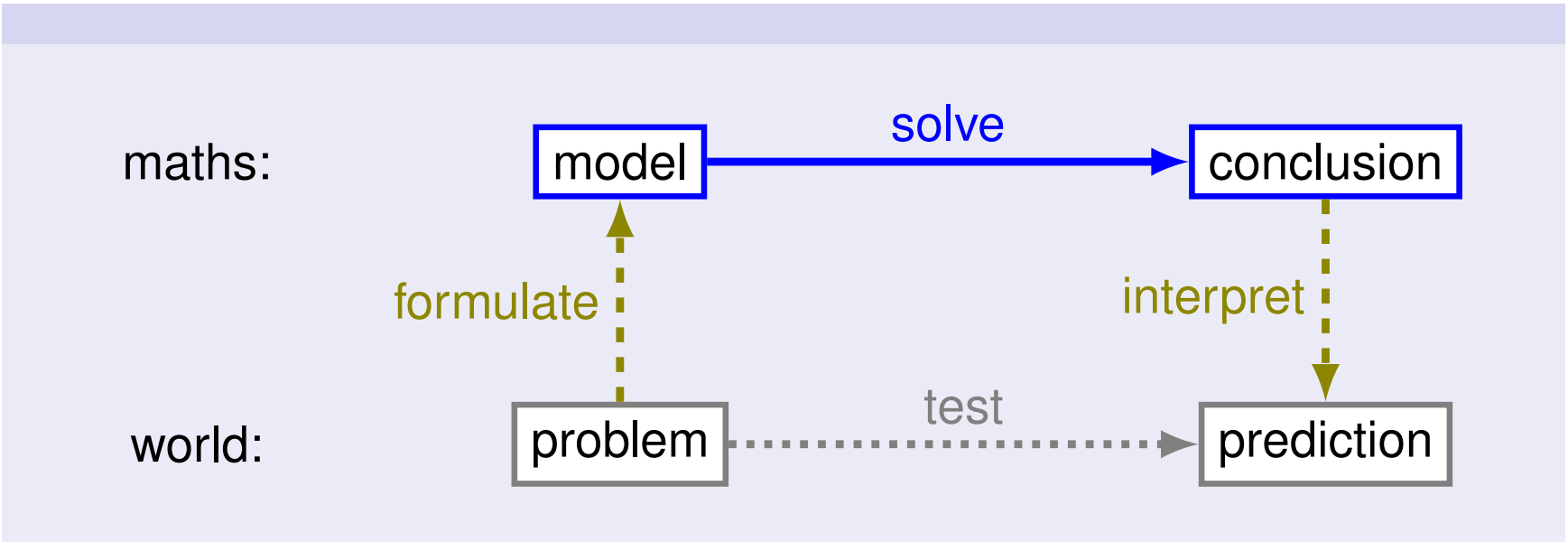


- The **area** A of a circle **depends** on the **radius** r of the circle.
- The **rule** that connects r and A is given by the **equation** $A = \pi r^2$. This **formula** assigns to each positive number r one value of A. We say that A **is a function of** r, and write
$$A(r) = \pi r^2.$$

Here, A is called a **dependent variable** and r is called an **independent variable**: A depends on r.

Mathematical Models

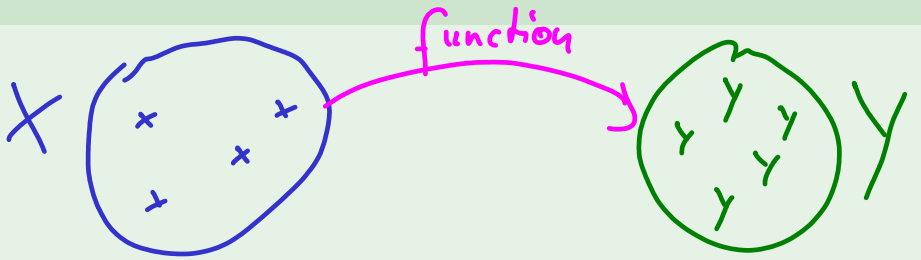
- A **mathematical model** is a **mathematical description** (by means of a function, equation) of a **real-world phenomenon**.
- The model helps to **understand** the phenomenon, and perhaps to make **predictions** about future behavior.



- A good model **simplifies reality** to permit **mathematical calculations**, while being **sufficiently accurate** to provide **valuable conclusions**.
- Be aware of the **limitations** of the chosen model.

Functions

A **function** is a **rule** that maps an element of one set to another unique element of another set.



We represent a function symbolically in two ways, either

$$f : x \to y$$

or

$$y = f(x)$$

Here x is in the set of X (or $x \in X$), and y is in the set of Y (or $y \in Y$).

Functions

The set X is called the **domain** of the function.
The set Y is called the **codomain**.

When $y = f(x)$ for some $x \in X$, y is said to be the **image** of x under f.

The set of all images $y = f(x), x \in X$, is called the **range** (or **image set**) of f.

Note: When we write $y = f(x)$, "x" is known as the **argument** of the function.

- Here, the sets X and Y are usually **sets of numbers**.
- It is not necessary for all elements y of the codomain Y to be images under f.
- One element $y \in Y$ can serve as value $f(x)$ for several $x \in X$.

4 Ways to Represent a Function

A function can be represented in many differents ways:

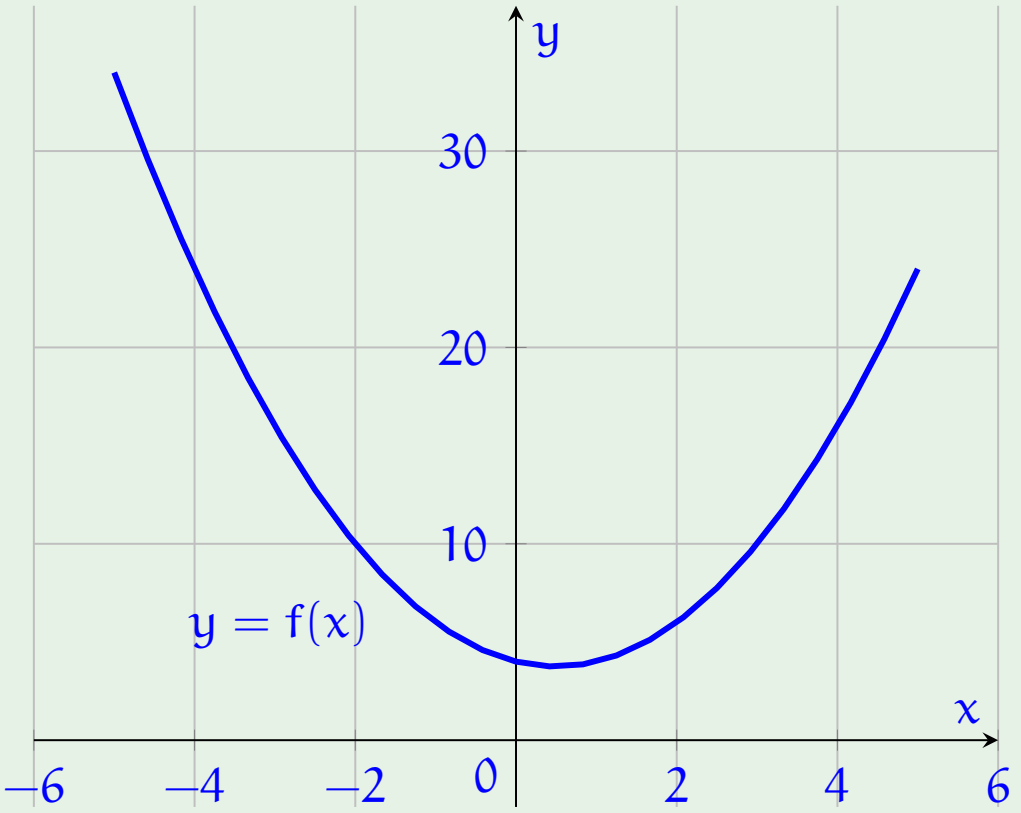
1. **verbally** (by a description in **words**);
2. **numerically** (as a **table** of values);
3. **visually** (as a **graph**);
4. **algebraically** (by an explicit **formula**).

Often it is possible, and useful, to go from one way to another.

Graphical Representation

Graph $\to$ Table

A common way to **visualize** a function $f\colon X \to \mathbb{R}$ is its **graph** in the x, y-plane.



x	$f(x)$
-4	24
-2	10
0	4
2	6
4	16

Domain Convention

Often, the domain of a function is not expilicitly stated.
In such a case the following **Domain Convention** applies.

The **domain** of a function f is the set of all numbers x for which $f(x)$ **makes sense** and gives a **real-number output**.

Example

1. Find the domain D of $f(x) = \frac{1}{x^2 - x}$.

division by zero !

Since $x^2 - x = x(x-1)$,

$g(x)$ is not defined when $x = 0$ or $x = 1$

So $D = \{x \mid x \neq 0, x \neq 1\} = (-\infty, 0) \cup (0, 1) \cup (1, \infty)$

2. Find the domain of the function $f(x) = \sqrt{x + 2}$.

$D = \{x \mid x \geq -2\}$

$= [-2, \infty)$

$\sqrt{x}$ is not real for negative x

Functions

Example

Identify the domain, codomain and range of

1. $f(x) = 3x^2 + 1$
2. $f(x) = \sqrt{(x+4)(3-x)}$
3. $g(x) = \frac{1}{x}$

Solution

(1) $f(x) = 3x^2 + 1$

- $f(x)$ can be evaluated for all $x \in \mathbb{R}$, so domain $= \mathbb{R}$
- The lowest value $f(x)$ can take is 1 (when $x = 0$), so range $= [1, \infty)$ or $\{y \mid y \geq 1, y \in \mathbb{R}\}$
- We could choose $\mathbb{R}$ as codomain as it contains the range.

Functions

Example

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Solution ctd.

(2) $f(x) = \sqrt{(x+4)(3-x)}$

- Domain is $[-4, 3]$. Outside this range the function is not real valued.

Why?

$(x+4)(3-x) \geq 0 \Rightarrow [x+4 \geq 0 \text{ and } 3-x \geq 0]$ or $[x+4 < 0 \text{ and } 3-x < 0]$

$\Rightarrow [x \geq -4 \text{ and } x \leq 3]$ $[x < -4 \text{ and } x > 3]$ not possible

So $-4 \leq x \leq 3$.

- Range ? ... We'll discuss this tomorrow.