Projections

Let U be a subspace of $\mathbb{R}^n$.

The *orthogonal complement* of U, denoted $U^\perp$, is the subspace of $\mathbb{R}^n$ consisting of *all* vectors in $\mathbb{R}^n$ that are orthogonal to *every* vector in U.

1. Note that $U^\perp$ really is a subspace.

Suppose that $\mathbf{v}, \mathbf{w} \in U^\perp$. Thus $\mathbf{v}.\mathbf{u} = \mathbf{w}.\mathbf{u} = 0$ for all $\mathbf{u} \in U$.

Then $(\mathbf{v} + \mathbf{w}).\mathbf{u} = \mathbf{v}.\mathbf{u} + \mathbf{w}.\mathbf{u} = 0 + 0 = 0$.

Also, if $r \in \mathbb{R}$, then $r\mathbf{v}.\mathbf{u} = r(\mathbf{v}.\mathbf{u}) = r.0 = 0$.

2. Secondly, note that $U \cap U^\perp = \{\mathbf{0}\}$. Let $\mathbf{v} \in U \cap U^\perp$, then $\mathbf{v}.\mathbf{v} = 0$. Now $\mathbf{v}.\mathbf{v} = ||\mathbf{v}||^2$ by the definition of dot product (check). Thus $||\mathbf{v}||^2 = 0$, i.e., $||\mathbf{v}|| = 0$, i.e., $\mathbf{v} = \mathbf{0}$.

Example. Let $U = \langle \mathbf{u} \rangle$ be the 1-d subspace of $\mathbb{R}^2$ spanned by $\mathbf{u} := (1,2)$; i.e., all scalar multiples of $(1,2)$. Find

(i) $U^{\perp}$;

(ii) $\mathbf{v}_1 \in U$ and $\mathbf{v}_2 \in U^{\perp}$ such that $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{v} = (5,3)$.

Solution. If $\mathbf{w} := (a,b) \in U^{\perp}$ then $\mathbf{w}.\mathbf{u} = 0 \Rightarrow a + 2b = 0$ $\Rightarrow a = -2b$.

Thus $\mathbf{w} = (-2b, b) = b(-2,1)$. So $U^{\perp} = \langle (-2,1) \rangle$; i.e., all scalar multiples of $(-2,1)$.

Must find $r, s \in \mathbb{R}$ such that $\mathbf{v}_1 = r(1,2)$ and $\mathbf{v}_2 = s(-2,1)$, i.e.,

$$(5,3) = r(1,2) + s(-2,1) = (r - 2s, 2r + s).$$

As a matrix equation: $X\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, where $X = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$.

Solution (continued).

Note $\det(X) = 5 \neq 0$, so X is invertible.

Thus $X\binom{r}{s} = \binom{5}{3} \Rightarrow \binom{r}{s} = X^{-1}\binom{5}{3} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\binom{5}{3} = \frac{1}{5}\begin{pmatrix} 11 \\ -7 \end{pmatrix}$.

That is, $r = 11/5$ and $s = -7/5$.

So $\mathbf{v}_1 = r(1,2) = (\frac{11}{5}, \frac{22}{5})$, $\mathbf{v}_2 = s(-2,1) = (\frac{14}{5}, -\frac{7}{5})$.

Check: $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_1.\mathbf{v}_2 = 0$. □

Returning to the general setting: given a subspace U of $\mathbb{R}^n$, the aims are to (i) find $U^\perp$, then (ii) describe all of $\mathbb{R}^n$ in terms of U and $U^\perp$.

The main ideas in achieving these aims are illustrated in the preceding example.

Suppose that $\dim(U) = k$ and U has basis $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$.

Let A be the matrix whose first row is $\mathbf{b}_1$, second row is $\mathbf{b}_2, \ldots, k$th row is $\mathbf{b}_k$. Thus A is a $k \times n$ matrix.

Definition. The *rank* of a matrix M, denoted $\operatorname{rank}(M)$, is the maximum number of the rows of M that are linearly independent. (There is a theorem that states that $\operatorname{rank}(M)$ is the maximum number of *columns* of M that are linearly independent, too.)

An $n \times n$ matrix is invertible if and only if it has rank n.

As the set of rows of A is linearly independent by definition, $\operatorname{rank}(A) = k$.

If $\mathbf{v} \in U^\perp$ then $\mathbf{v}.\mathbf{b}_i = 0$ for all i; so $A\mathbf{v}^\top = \mathbf{0}$. Conversely, if $A\mathbf{v}^\top = \mathbf{0}$ then $\mathbf{v}.\mathbf{b}_i = 0$ for all i. Thus $\mathbf{v}.\mathbf{w} = 0$ for all $\mathbf{w} \in U$, as the $\mathbf{b}_i$ span U ($\mathbf{v}$ is orthogonal to every linear combination of $\mathbf{b}_i$s).

This shows that $U^\perp$ is the *nullspace* or *kernel* of A, i.e., the subspace of all $\mathbf{v} \in \mathbb{R}^n$ such that $A\mathbf{v}^\top = \mathbf{0}$. Dimension of the nullspace is called the *nullity*.

Now the "rank-nullity theorem" states that

$$\boxed{\mathrm{rank}(A) + \text{nullity of } A = \text{no. columns of } A}$$

which implies that $k + \dim(U^\perp) = n \Rightarrow \boxed{\dim(U^\perp) = n - k}$.

Hence, there are $\mathbf{b}_{k+1}, \ldots, \mathbf{b}_n \in \mathbb{R}^n$ that form a basis of $U^\perp$.

Claim: the n vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n$ (i.e., basis of U $\cup$ basis of $U^\perp$) are linearly independent.

To verify this claim, suppose that $a_1, \dots, a_n \in \mathbb{R}$ and

$$a_1\mathbf{b}_1 + \cdots + a_k\mathbf{b}_k + a_{k+1}\mathbf{b}_{k+1} + \cdots + a_n\mathbf{b}_n = \mathbf{0}$$

$$a_1\mathbf{b}_1 + \cdots + a_k\mathbf{b}_k = -(a_{k+1}\mathbf{b}_{k+1} + \cdots + a_n\mathbf{b}_n).$$

The left-hand vector in this equation is in U; the right-hand vector is in $U^\perp$.

Recall that $U \cap U^\perp = \{\mathbf{0}\}$ (point 2. at the beginning of this lecture).

Thus $a_1\mathbf{b}_1 + \cdots + a_k\mathbf{b}_k = \mathbf{0} \Rightarrow a_1 = \cdots = a_k = 0$ as the $\mathbf{b}_1, \dots,$ $\mathbf{b}_k$ are linearly independent. Likewise, $a_{k+1} = \cdots = a_n = 0$.

We have therefore proved that $\{$basis of $U\} \cup \{$basis of $U^\perp\}$ is linearly independent. Since this set has size n, it is a basis of $\mathbb{R}^n$.

In summary: given any $\mathbf{w} \in \mathbb{R}^n$, there are $\mathbf{u} \in U$ and $\mathbf{v} \in U^\perp$ such that

$$\mathbf{w} = \mathbf{u} + \mathbf{v}.$$

Moreover, $\mathbf{u}$ & $\mathbf{v}$ are unique with these properties, for the given $\mathbf{w}$. They are called, respectively, *the projection of* $\mathbf{w}$ *on* U, and *the projection of* $\mathbf{w}$ *on* $U^\perp$.

Write $\mathbb{R}^n = U \oplus U^\perp$, and say "$\mathbb{R}^n$ is the direct sum of the subspaces U and $U^\perp$".

Question: how can we find the projection of a given vector in $\mathbb{R}^n$ on a subspace and its orthogonal complement?