

Basic linear algebra terminology

We now define (recall?) some basic, standard terms:

- linear combination
 - subspace
 - (linear) span
 - linear independence, linear dependence
 - basis
 - dimension.
- Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$. A *linear combination* of these vectors is a vector of the form

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k, \quad a_i \in \mathbb{R}.$$

- A (non-empty) subset U of \mathbb{R}^n is a *subspace* of \mathbb{R}^n if every linear combination of the elements of U is an element of U .

In other words, a subspace is *closed* under all linear combinations. Every subspace must contain the zero vector $\mathbf{0} := (0, 0, \dots, 0)$. (Why?)

e.g., $S = \{(a, b) \in \mathbb{R}^2 \mid a + b = 1\}$ is *not* a subspace of \mathbb{R}^2 : doesn't contain $(0, 0)$.

We can also check that some linear combinations of elements of S aren't in S ; say $(1, 0), (0, 1) \in S$ but $(1, 1) = (1, 0) + (0, 1) \notin S$. S defines a straight line in \mathbb{R}^2 not passing through the origin.

e.g., $S = \{(a, b) \in \mathbb{R}^2 \mid b = 2a\}$ is a subspace of \mathbb{R}^2 (check).

- Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\} \subset \mathbb{R}^n$. The (*linear*) *span* of S , denoted $\text{span}(S)$, is the set of all linear combinations of the vectors in S :

$$\{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k \mid a_i \in \mathbb{R}\}.$$

$\text{span}(S)$ is a subspace of \mathbb{R}^n .

- $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\} \subset \mathbb{R}^n$ is *linearly independent* if, whenever there are scalars $a_1, \dots, a_k \in \mathbb{R}$ such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$$

then all of the scalars a_1, \dots, a_k are zero. In other words, no \mathbf{v}_i in a linearly independent set can be expressed as a linear combination of the remaining vectors in the set (equivalent definition).

If S is not linearly independent, then it is said to be *linearly dependent*—one or more vectors in S are redundant in $\text{span}(S)$.

Example. Let $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (4, 5, 6)$, $\mathbf{v}_3 = (7, 8, 9)$ in \mathbb{R}^3 .

Then

$$-\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 = (-1, -2, -3) + (8, 10, 12) + (-7, -8, -9) = (-1 + 8 - 7, -2 + 10 - 8, -3 + 12 - 9) = (0, 0, 0).$$

Hence \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 are linearly dependent.

Example. Are $\mathbf{v}_1 = (7, 0, 4, 0, 1, 9, 0)$, $\mathbf{v}_2 = (6, 0, 7, 1, 4, 8, 0)$, $\mathbf{v}_3 = (5, 0, 6, 2, 3, 1, 7)$, $\mathbf{v}_4 = (4, 5, 3, 3, 2, 2, 4)$ linearly independent in \mathbb{R}^7 ?

Solution. Yes. For suppose that $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 = \mathbf{0}$. Looking at second coordinates, $0 + 0 + 0 + 5a_4 = 0 \Rightarrow a_4 = 0$. Then looking at seventh coordinates, $0 + 0 + 7a_3 = 0 \Rightarrow a_3 = 0$. Then looking at fourth coordinates, $0 + a_2 = 0 \Rightarrow a_2 = 0$. Hence $a_1 = 0$ too.

- Let U be a subspace of \mathbb{R}^n . A finite subset B of U is a *basis* of U if (i) B is linearly independent; (ii) $\text{span}(B) = U$.

If B is a basis of U , then each element in U can be expressed as a unique linear combination of the elements of B (check).

Theorem

Let U be a subspace of \mathbb{R}^n .

- (i) **EXISTENCE** U has a basis.
- (ii) **UNIQUENESS** Any two bases of U have the same number of elements.

The unique number in (ii) of the theorem is called the *dimension* of U , denoted $\dim(U)$.

Note:

- $\dim(\mathbb{R}^n) = n$; e.g., $\{(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, 0, \dots, 0, 1)\}$ is a basis of \mathbb{R}^n (check that these n vectors are linearly independent, and that they span \mathbb{R}^n).
- Let U be a subspace of \mathbb{R}^n . Then $\dim(U) \leq n$. Also, $\dim(U) = n$ if and only if $U = \mathbb{R}^n$.

Example. If U is a subspace of \mathbb{R}^2 , then $U = \{\mathbf{0}\}$, or $U = \mathbb{R}^2$, or the elements of U lie on a single straight line through $(0, 0)$.

Theorem

- (i) *By adding vectors if needed, any linearly independent subset of \mathbb{R}^n can be extended to a basis of \mathbb{R}^n .*
- (ii) *By deleting vectors if needed, any spanning set of \mathbb{R}^n can be reduced to a basis of \mathbb{R}^n .*

As consequences of the preceding theorem: any n linearly independent vectors in \mathbb{R}^n form a basis of \mathbb{R}^n ; any n spanning vectors in \mathbb{R}^n form a basis of \mathbb{R}^n .

Example. $\mathbf{u} = (1, 2)$ and $\mathbf{v} = (3, 4)$ form a basis of \mathbb{R}^2 .

Proof. Let $(r, s) \in \mathbb{R}^2$. We show that there exist $a, b \in \mathbb{R}$ such that $(r, s) = a\mathbf{u} + b\mathbf{v} = a(1, 2) + b(3, 4) = (a + 3b, 2a + 4b)$.

As a matrix equation this is $X \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} r \\ s \end{pmatrix}$, where $X = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

Now $\det(X) = 4 \cdot 1 - 3 \cdot 2 = -2 \neq 0$, so X^{-1} exists. (Indeed, $X^{-1} = \frac{1}{2} \begin{pmatrix} -4 & 3 \\ 2 & -1 \end{pmatrix}$; check.)

Hence $\begin{pmatrix} a \\ b \end{pmatrix} = X^{-1}X \begin{pmatrix} a \\ b \end{pmatrix} = X^{-1} \begin{pmatrix} r \\ s \end{pmatrix}$.

Proof (continued).

This shows that a, b can be calculated for any r, s .

$$\text{(Check that } \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} (3s - 4r)/2 \\ (2r - s)/2 \end{pmatrix}.)$$

Thus \mathbf{u}, \mathbf{v} span \mathbb{R}^2 .

As there are 2 vectors in the spanning set, and $\dim(\mathbb{R}^2) = 2$, it follows that $\{\mathbf{u}, \mathbf{v}\}$ is a basis of \mathbb{R}^2 .

The above proof also shows that \mathbf{u} and \mathbf{v} are linearly independent: if $(r, s) = (0, 0)$ then $a = b = 0$.