MA2102 Mathematics

Lecturer (first half): Mike Welby

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This course covers some topics in mathematics that are indispensible in engineering:

- linear algebra (orthogonality, eigenvalues and eigenvectors),
- infinite sequences and series,
- calculus with complex numbers.

Prerequisites are the three semesters of mathematics covered to now.

Blackboard is the primary resource for the module. Announcements, lecture material, assessment, grades etc. can be found there.

Tutorials: TBA.

Assessment: $\frac{1}{3}$ Continuous (WeBWorKs), $\frac{2}{3}$ Final Exam.

Suggested references, weeks 1–6.

- Modern Engineering Mathematics, G. James, Prentice Hall (510.2462 MOD);
- Advanced Modern Engineering Mathematics, G. James, Prentice Hall (on library website);
- Advanced Engineering Mathematics, E. Kreyszig, Wiley (510.2462 KRE)
- Online: many!

Orthogonality

Recall that $\mathbb{R}^n$ is the n-dimensional vector space over the real numbers $\mathbb{R}$ (*n-dimensional Euclidean space*).

Elements of $\mathbb{R}^n$ are *vectors*: n-tuples $(a_1, \dots, a_n)$ of real numbers a_i.
We generally use boldface to denote vectors. Elements of $\mathbb{R}^1 = \mathbb{R}$ are called *scalars*.

Basic operations in $\mathbb{R}^n$ are *addition* and *scalar multiplication*. These are done componentwise: if $\mathbf{u} = (u_1, \dots, u_n)$, $\mathbf{v} = (v_1, \dots, v_n)$, and $\lambda \in \mathbb{R}$, then

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, \dots, u_n + v_n),$$
$$\lambda\mathbf{u} = (\lambda u_1, \dots, \lambda u_n).$$

Example. $(-1, 0, 3) + (2, 5, 7) = (1, 5, 10)$, $\frac{1}{2}(9, 12) = (4.5, 6)$.

Note that the operations are *closed*, i.e., the sum of any number of vectors in $\mathbb{R}^n$ is a vector in $\mathbb{R}^n$; the scalar multiple of a vector in $\mathbb{R}^n$ is a vector in $\mathbb{R}^n$.

Another operation, that sends pairs of vectors to scalars, is the *dot product*.

If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ then

$$\mathbf{u}.\mathbf{v} = \mathbf{u}\mathbf{v}^\top$$

where $^\top$ denotes *transpose* (write rows of a given matrix as columns, and vice versa). Note $\mathbf{u}.\mathbf{v} = \mathbf{v}.\mathbf{u}$ (*why*?).

Coordinatewise definition of dot product: if $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ then

$$\mathbf{u}.\mathbf{v} = u_1 v_1 + \cdots + u_n v_n.$$

e.g., $(3,2,-5).(1,11,4) = (3,2,-5)\begin{pmatrix} 1 \\ 11 \\ 4 \end{pmatrix} = 3.1 + 2.11 + (-5)4 = 3 + 22 - 20 = 5.$

Remember that $\mathbf{u} \in \mathbb{R}^n$ is completely determined by its *length* and *direction*.

Length $||\mathbf{u}||$ of $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$ is the length of the line segment connecting the *point* $(u_1, \ldots, u_n)$ to the origin $(0, \ldots, 0)$ in $\mathbb{R}^n$. By Pythagoras' theorem, this is

$$||\mathbf{u}|| = \sqrt{u_1^2 + \cdots + u_n^2}$$

If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ then

$$\mathbf{u}.\mathbf{v} = ||\mathbf{u}||\,||\mathbf{v}||\,\cos\theta \qquad (*)$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. Follows from the Cosine Rule in trigonometry.

We say that $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are *orthogonal* if their dot product is zero.

If $\mathbf{u}, \mathbf{v}$ are orthogonal then we write $\mathbf{u} \perp \mathbf{v}$.

For non-zero $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^2$, we have $\mathbf{u} \perp \mathbf{v}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ as line segments are orthogonal/perpendicular/at right angles: by $(*)$, $\cos\theta = 0$ so $\theta = \frac{\pi}{2}$.

Example. Find all vectors in $\mathbb{R}^4$ that are orthogonal to both $\mathbf{u}_1 = (3, 4, 3, 2)$ and $\mathbf{u}_2 = (1, 2, 1, 2)$.

Solution. Let $\mathbf{v} = (a, b, c, d) \in \mathbb{R}^4$ and suppose that $\mathbf{u}_1.\mathbf{v} = \mathbf{u}_2.\mathbf{v} = 0$.

As a single matrix equation this is $\begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} \mathbf{v}^\top = \mathbf{0}$.

Solution (continued). That is, $\begin{pmatrix} 3 & 4 & 3 & 2 \\ 1 & 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

To solve this (homogeneous) system, we reduce the l.h. matrix to row-echelon form:

$$\begin{pmatrix} 3 & 4 & 3 & 2 \\ 1 & 2 & 1 & 2 \end{pmatrix} \; {\scriptstyle R_1 \leftrightarrow R_2} \rightsquigarrow \begin{pmatrix} 1 & 2 & 1 & 2 \\ 3 & 4 & 3 & 2 \end{pmatrix}$$

$$ {\scriptstyle R_2 \leftarrow R_2 - 3R_1} \rightsquigarrow \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & -2 & 0 & -4 \end{pmatrix}$$

$$ {\scriptstyle R_2 \leftarrow -\frac{1}{2} R_2} \rightsquigarrow \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

$$ {\scriptstyle R_1 \leftarrow R_1 - 2R_2} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 2 \end{pmatrix}.$$

Solution (continued).

Thus $a + c - 2d = 0$ (first row of the echelon form matrix) and $b + 2d = 0$ (second row of the echelon form matrix), i.e., $a = -c + 2d$ and $b = -2d$.

Hence $\mathbf{v} = (-c + 2d, -2d, c, d) = c(-1, 0, 1, 0) + d(2, -2, 0, 1)$.

The set of *all* vectors orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$ is therefore the *linear span*

$$\{c(-1, 0, 1, 0) + d(2, -2, 0, 1) \mid c, d \in \mathbb{R}\}$$

of $(-1, 0, 1, 0)$ and $(2, -2, 0, 1)$.