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- Fundamental Units are

MASS, LENGTH, TIME

All physical quantities can be obtained by combining the fundamental units.

- Notation: ϕ physical quantity $\Rightarrow [\phi]$ its dimensions

We will denote by:

M = dimension of mass (1kg,... in the SI int. sys. units)
L = dimension of length (1m in the SI)
T = dimension of time

- Derived Quantities:

Examples: $[\text{area}] = L^2$
$[\text{volume}] = L^3$
$[\text{velocity}] = LT^{-1}$
$[\text{acceleration}] = LT^{-2}$
$[\text{force}] = MLT^{-2}$ ($F = ma$)
$[\text{work}] = ML^2T^{-2}$ ($W = \underline{F} \cdot \underline{s}$)
$[\text{power}] = ML^2T^{-3}$ ($\mathcal{P} = dW/dt$)

Thus we see that every derived variable appears in the form

$$[Q] = M^a L^b T^c \quad (\text{say } [\text{area}] = M^0 L^2 T^0)$$

↑ any physical quantity

- Important definition:

Π is a DIMENSIONLESS variable if $[\Pi] = M^0 L^0 T^0 = 1$

↑ Ps it's the capital letter π of the Greek alphabet

Example of dimensionless variables:

- Strain: $\varepsilon = \dfrac{\Delta \ell}{\ell} = \dfrac{\text{variation of length}}{\text{length}}$ $\quad [\varepsilon] = 1$
- Angles: $\vartheta = \dfrac{s}{r} = \dfrac{\text{arclength}}{\text{radius}}$ $\quad [\vartheta] = 1$

Buckingham's Theorem (also known as Π-theorem)

Physical Problem ----> Suppose it's governed by a physical law

$$F(Q_1, \dots, Q_n) = 0 \qquad (*)$$

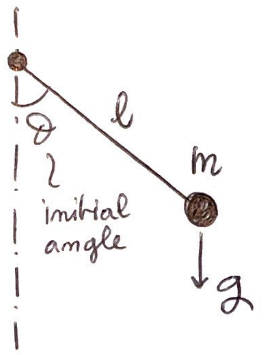
where $Q_1, \dots, Q_n$ are the involved variables.

Then denote by k the number of Physical dimensions involved in the problem. Π-theorem then says that you can reformulate $(*)$ as

$$\hat{F}(\Pi_1, \dots, \Pi_p) = 0 \qquad p = n - k \leqslant n$$

Where $\Pi_1, \dots, \Pi_p$ are DIMENSIONLESS VARIABLES.

- Example: Simple Pendulum. Find the period τ of the oscill.



- involved variables: the Q_i

$(\tau, \ell, g, m, \vartheta)$ $\quad \boxed{n = 5}$

- fundamental physical dimensions:

$[\tau] = T$
$[\ell] = L$
$[g] = LT^{-2}$
$[m] = M$
$[\vartheta] = 1$

we see that all M, L, T are involved, thus $\boxed{k = 3}$

Then Π-Theorem secures that a law:

$$F(\tau, \ell, m, g, \vartheta) = 0$$

may be EQUIVALENTLY recast as

$$\hat{F}(\Pi_1, \Pi_2) = 0 \qquad (\text{since } p = n - k = 5 - 3 = 2)$$

in terms of only TWO dimensionless variables Π_1, Π_2.

How to find Π_1, Π_2? Assume that

$$\Pi = \tau^a \ell^b g^c m^d \vartheta^e$$

and impose $[\Pi] = 1$. So

$$[\Pi] = T^a L^b (LT^{-2})^c M^d \overbrace{1^e}^{\text{it's already dimless!}} = M^d L^{b+c} T^{a-2c} = M^0 L^0 T^0$$

we get the system:

$$\begin{cases} d = 0 \\ b + c = 0 \\ a - 2c = 0 \end{cases}$$

3 EQUATIONS for **4** UNKNOWNS
it would admit $\infty^{(4-3)} = \infty^1$ solutions

Take for example c as a parameter (that is, treat c as if it's fixed) then

$$\begin{cases} b = -c \\ a = 2c \end{cases} \Rightarrow \Pi = \tau^{2c} \ell^{-c} g^c m^0 \vartheta^e = \left(\frac{\tau^2 g}{\ell}\right)^c \vartheta^e$$

Since at this point we know there are 2 dimless var, we thus obtain them by setting:

$$\begin{cases} c = 1,\ e = 0 & : \Pi_1 = \dfrac{\tau^2 g}{\ell} \\ c = 0,\ e = 1 & : \Pi_2 = \vartheta \end{cases}$$

(CHECK THEY ARE DIMENSIONLESS!)

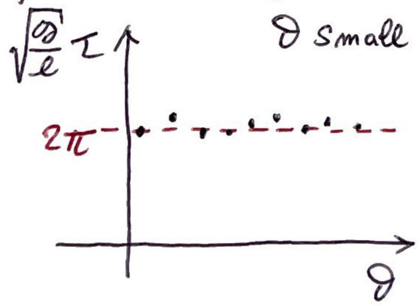
Finally, we obtain that the physical law may be recast as

$$(**) \quad \hat{F}(\Pi_1, \Pi_2) = 0 \Rightarrow \hat{F}\left(\frac{\tau^2 g}{\ell}, \vartheta\right) = 0.$$

To further reduce the problem, assume we can INVERT (**) that is to rewrite it as:

$$\underbrace{\Pi_1 = G(\Pi_2)}_{\text{just some function}} \Rightarrow \frac{\tau^2 g}{\ell} = G(\vartheta) \Rightarrow \tau = \sqrt{G(\vartheta)}\sqrt{\frac{\ell}{g}}$$

To find $G(\vartheta)$ you can do EXPERIMENTS

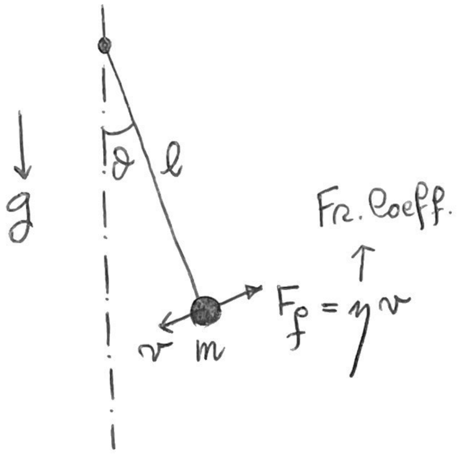


which is precisely what the theory says!

$$\tau = 2\pi\sqrt{\frac{\ell}{g}}$$

2 Ps DO IT AT HOME! it's easy!

•) Simple Pendulum WITH FRICTION



With respect to the problem already seen there is a new variable:

$$\eta = \text{Friction Coefficient} = \frac{F_f}{v}$$

Dimensionally:

$$[\eta] = \frac{[F]}{[v]} = \frac{MLT^{-2}}{LT^{-1}} = MT^{-1}$$

- $n = 6$ since $\{\tau, \ell, g, m, \vartheta, \eta\}$ = involved variables
- $r = 3$ (Clearly as already seen in the prev. ex.)

└ Π-theorem says there is a law: $\hat{F}(\Pi_1, \Pi_2, \Pi_3) = 0$
└ $p = n - r = 3$.

- To find the dimless variables:

Assume: $\Pi = \tau^a \ell^b g^c m^d \vartheta^e \eta^f$

Impose $[\Pi] = M^0 L^0 T^0 =$

$$= [\tau]^a [\ell]^b [g]^c [m]^d [\vartheta]^e [\eta]^f$$
$$= T^a L^b L^c T^{-2c} M^d 1^e M^f T^{-f}$$
$$= M^{(d+f)} L^{(b+c)} T^{(a-2c-f)}$$

Thus we get the system: (REMARK: e drops out of the system)

$$\begin{cases} d + f = 0 \\ b + c = 0 \\ a - 2c = f \end{cases}$$

5 unknowns in 3 equations: ∞^2 solutions
└ WE NOW HAVE TO SELECT **2** PARAM, say (c, d)

↓

THUS:
$$\begin{cases} f = -d \\ b = -c \\ a = 2c - d \end{cases}$$

So

$$\Pi = \tau^{2c} \ell^{-c} g^c m^d \vartheta^e \eta^{-d} \tau^{-d} = \left(\frac{\tau^2 g}{\ell}\right)^c \left(\frac{m}{\eta\tau}\right)^d \vartheta^e$$

Thus the 3 dimensionless v. are:
$$\begin{cases} c = 1,\ d = e = 0 & \Pi_1 = \dfrac{\tau^2 g}{\ell} \\ c = 0 = e,\ d = 1 & \Pi_2 = m/(\tau\eta) \\ c = 0 = d,\ e = 1 & \Pi_3 = \vartheta \end{cases}$$

So we have by Buckingham's theorem:

$$\hat{F}\Big(\underbrace{\frac{\tau^2 g}{\ell}}_{\Pi_1}, \underbrace{\frac{m}{\eta\tau}}_{\Pi_2}, \underbrace{\vartheta}_{\Pi_3}\Big) = 0$$

WHAT IF I WANT TO FIND HOW τ DEPENDS ON THE OTHER V?

└ Combining dimensionless variables leads always to dimensionless variables! Let us use this fact to create a new set $(\bar{\Pi}_1, \bar{\Pi}_2, \bar{\Pi}_3)$ where τ is featured only in one variable:

$$\begin{cases} \Pi_1 = \tau^2 \dfrac{g}{\ell} \\ \Pi_2 = \dfrac{m}{\eta\tau} \end{cases} \qquad \Rightarrow \tau = \sqrt{\frac{\ell}{g}}\,\Pi_1^{1/2} \qquad \text{SET } \bar{\Pi}_1 = \Pi_1 \ \& \ \bar{\Pi}_3 = \Pi_3$$

$$\Pi_2 = \frac{m}{\eta}\sqrt{\frac{g}{\ell}}\,\Pi_1^{-1/2} \Rightarrow \text{DEFINE } \bar{\Pi}_2 = \Pi_2 \Pi_1^{1/2} = \frac{m}{\eta}\sqrt{\frac{g}{\ell}}$$

Now: by Buckingham's theorem there will be ANOTHER

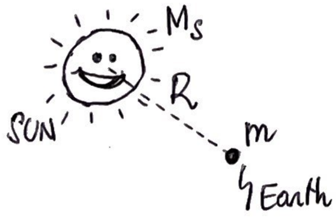
$$\bar{F}(\bar{\Pi}_1, \bar{\Pi}_2, \bar{\Pi}_3) = 0$$

$$\bar{\Pi}_1 \to \tau^2\frac{g}{\ell}, \quad \bar{\Pi}_2 \to \frac{m}{\eta}\sqrt{\frac{g}{\ell}}, \quad \bar{\Pi}_3 \to \vartheta$$

So to find τ we assume $\bar{F}$ can be inverted:

$$\bar{\Pi}_1 = G(\bar{\Pi}_2, \bar{\Pi}_3) \Rightarrow \tau = \sqrt{\frac{\ell}{g}}\underbrace{\sqrt{G\left(\frac{m}{\eta}\sqrt{\frac{g}{\ell}}, \vartheta\right)}}_{\text{correction due to initial amplitude \& air viscosity}}$$

• **KEPLER'S LAW** - find the period of revolution of the Earth around the Sun



$m \ll M$ we can neglect it

Variables: (τ, M, R, G) $n = 4$

G = universal constant of gravitation

└ $[G]$ given by Newton's law:

$$F = G \cdot \frac{m_1 m_2}{r^2} \Rightarrow [G] = [F][r]^2[m_1]^{-1}[m_2]^{-1}$$

$$[G] = MLT^{-2} \cdot L^2 \cdot M^{-1} M^{-1} = M^{-1} L^3 T^{-2}$$

$$\begin{cases} [\tau] = T \\ [M] = M \\ [R] = L \\ [G] = M^{-1} L^{3} T^{-2} \end{cases} \Rightarrow r = 3 \text{ (3 fund. units define al vars.)}$$

THERE IS ONLY $1 = n - r = 4 - 3$ DIMENSIONLESS V.

to find it:

$$\Pi = M^a R^b \tau^c G^d \Rightarrow \text{impose } [\Pi] = M^0 L^0 T^0$$

$$[\Pi] = M^a L^b (T^c)(M^{-1} L^3 T^{-2})^d = M^{(a-d)} L^{(b+3d)} T^{(c-2d)}$$

we get the system:

$$\begin{cases} a - d = 0 \\ b + 3d = 0 \\ c - 2d = 0 \end{cases}$$

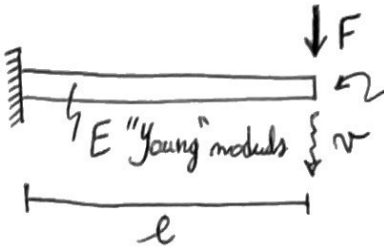
4 unknowns × 3 equations ∞^1 solutions – take $\underline{\underline{d}}$ as param:

$$\begin{cases} a = d \\ b = -3d \\ c = 2d \end{cases} \qquad \Pi = \left(\frac{M\tau^2 G}{R^3}\right)^d \quad \text{set } d = 1 \quad \Pi = \frac{M\tau^2 G}{R^3}$$

we obtain that $\tau^2 = \frac{\Pi}{MG} R^3 \Rightarrow$ THE SQUARE OF THE PERIOD IS PROPORTIONAL TO THE CUBE OF THE DISTANCE.

Exceptional Cases: WHEN $r \neq 3$!

•) **Cantilever Problem**: find the displacement $v \simeq \dfrac{F\ell^3}{EJ}$



circular cross section R

$J = \dfrac{\pi R^4}{4}$ "bending" moment of inertia

- $n = 5$ since involved variables are (v, F, ℓ, E, J)
- $r = ?$

$[F] = MLT^{-2}$ -----> SET $X = MT^{-2}$ ----> $[F] = XL$

$[v] = L$

$[E] = [\text{force}]/[\text{area}] = \dfrac{MLT^{-2}}{L^2} = ML^{-1}T^{-2}$ ---> $[E] = XL^{-1}$

$[\ell] = L$

$[J] = L^4$

thus we see that only A COMBINATION of M & T, $X = MT^{-2}$, plays a role $\Rightarrow r = 2 \Rightarrow$ there are $p = n - r = 5 - 2 = 3$ DIM.LESS V.

- to find the Π, assume $\Pi = F^a v^b E^c \ell^d J^e$ thus

$$[\Pi] = [F^a v^b E^c \ell^d J^e] = (XL)^a L^b (XL^{-1})^c L^d L^{4e}$$
$$= X^{(a+c)} L^{(a+b-c+d+4e)} = X^0 L^0$$

So we get the system of 2 EQUATIONS for **5** UNKNOWNS

$$\begin{cases} a + c = 0 \\ a + b - c + d + 4e = 0 \end{cases}$$

$5 - 2 = 3 \Rightarrow \infty^3$ solutions $\Rightarrow$ SET $\overset{\downarrow}{3}$ PARAMETERS: say b, c, e

then

$$\begin{cases} a = -c \\ d = 2c - b - 4e \end{cases} \Rightarrow \Pi = F^{-c} v^b E^c \ell^{2c-b-4e} J^e = \left(\frac{E\ell^2}{F}\right)^c \left(\frac{v}{\ell}\right)^b \left(\frac{J}{\ell^4}\right)^e$$

$$\begin{cases} \Pi_1 = \dfrac{E\ell^2}{F} \\ \Pi_2 = v/\ell \\ \Pi_3 = J/\ell^4 \end{cases}$$

thus by Buckingham's theorem we get:

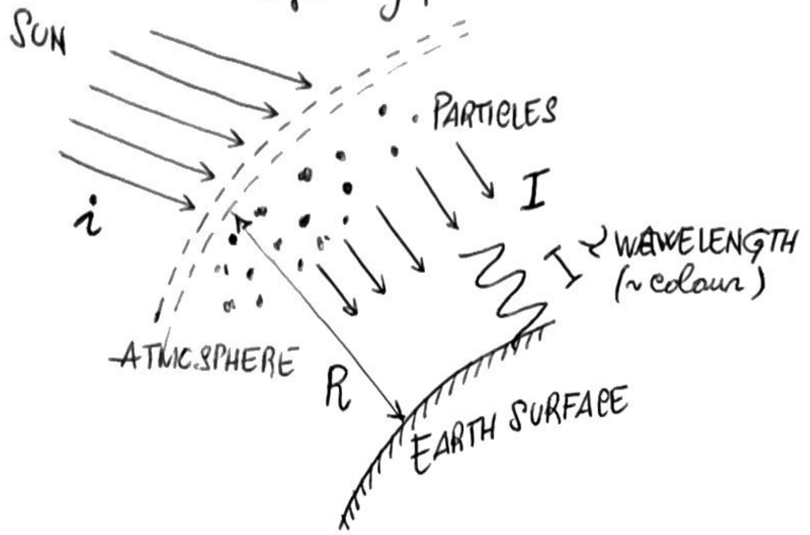
$$\hat{F}\left(\frac{E\ell^2}{F}, \frac{v}{\ell}, \frac{J}{\ell^4}\right) = 0 \Rightarrow \frac{v}{\ell} = G\left(\frac{E\ell^2}{F}, \frac{J}{\ell^4}\right).$$

In fact if we take a simple expression: $\dfrac{v}{\ell} = \dfrac{F}{E\ell^2} \cdot \dfrac{\ell^4}{J} \Rightarrow v \simeq \dfrac{F\ell^3}{EJ}$.

Give for home: Scattering problem (similar to the previous)

•) <u>WHY IS THE SKY BLUE ?</u> (Also this is a case where $r = 2$!)

The apparent colour of the sky is due to scattering of sunlight by particles in the atmosphere of the Earth



I = intensity of light after scattering

i = intensity of light before scattering

λ = wavelength

V = volume of particles

R = distance from the earth's crust.

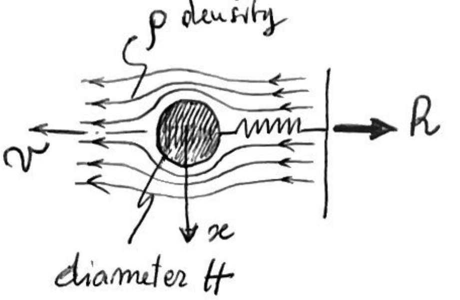
Find how λ depends on the involved variables.

<u>PS</u> $[i] = [I] = [\text{energy}][\text{area}]^{-1}[\text{time}]^{-1}$

$[\lambda] = L$

As in the previous problem you will find that $r=2$!

•) <u>Problem of motion in a viscous fluid</u> (Reynolds number)



"dynamic viscosity" relates:

$$\underbrace{\frac{R}{A}}_{\substack{\text{force on the object}\\\text{divided by area } A}} = \underbrace{\mu}_{\substack{\mu\\\text{dynamic viscosity}}} \underbrace{\frac{dv}{dx}}_{\text{gradient of velocity}}$$

Find a dimensionless relation among (R, μ, ρ, H, v) $n = 5$

Buckingham's theorem says:

$$F(R, \mu, \rho, H, v) = 0 \Leftrightarrow \hat{F}(\Pi_1, \Pi_2, \dots, \Pi_p) = 0$$

$p = n - r$

- find r:

$$\begin{cases} [\mu] = \dfrac{[R/A]}{[v/L]} = \dfrac{MLT^{-2} \cdot L^{-1}}{LT^{-1} \cdot L^{-1}} = ML^{-1}T^{-1} \\ [R] = MLT^{-2} \\ [H] = L \\ [\rho] = ML^{-3} \\ [v] = LT^{-1} \end{cases}$$

thus $r = 3 \Rightarrow p = n - r = 5 - 3 = 2$

there are 2 dimensionless v.

- find Πs: $\Pi = R^a H^b \rho^c \mu^d v^e$

$$[\Pi] = M^0 L^0 T^0 = [R]^a [H]^b [\rho]^c [\mu]^d [v]^e = \dots$$
$$= M^{(a+c+d)} L^{(a+b-3c-d+e)} T^{(-2a-d-e)}$$

hence the system:

$$\begin{cases} a+c+d=0 \\ a+b-3c-d+e=0 \\ -2a-d-e=0 \end{cases}$$

5 unknowns × 3 equations

∞^2 solutions $\Rightarrow$ set 2 parameters (a,d)

$$\downarrow$$

$$\dots \quad \begin{cases} b=-2a-d \\ c=-a-d \\ e=-2a-d \end{cases} \Rightarrow \Pi = \left(\frac{R}{H^2 \rho v^2}\right)^a \left(\frac{\mu}{H\rho v}\right)^d$$

thus $\Pi_1 = \dfrac{R}{H^2 \rho v^2}$; $\Pi_2 = \dfrac{\mu}{H \rho v}$

"Drag Coefficient" "Reynolds number"

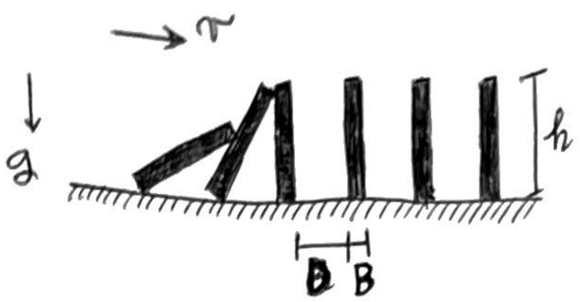
(actually the inverse: $Re = \dfrac{H\rho v}{\mu}$)

So $\hat{F}(\Pi_1, \Pi_2) = 0$ by imposing invertibility:

$$\Pi_1 = G(\Pi_2) \Rightarrow \frac{R}{H^2 \rho v^2} = G\left(\frac{\mu}{H\rho v}\right).$$

it defines the transition between LAMINAR & TURBULENT flow.

•) Toppling Dominos Problem:



Find how velocity v depends on the remaining variables

$F(v, D, B, h, g) = 0 \quad n = 5$

• find v ? $\begin{cases} [v] = L T^{-1} \\ [g] = L T^{-2} \\ [D] = [h] = [B] = L \end{cases}$

• Construct dimensionless variables:

$$\Pi = v^a D^b B^c h^d g^e \Rightarrow \text{impose } [\Pi] = M^0 L^0 T^0$$

thus $[\Pi] = \ldots = L^{(a+b+c+d+e)} T^{(-a-2e)}$

Thus $\begin{cases} a+b+c+d+e = 0 \\ a + 2e = 0 \end{cases}$ 5 unknowns × 2 equs

$p = n - r = 3$ dimless. variables

• Thus $\Pi = v^{(-2e)} d^{(e-c-d)} B^c h^d g^e =$

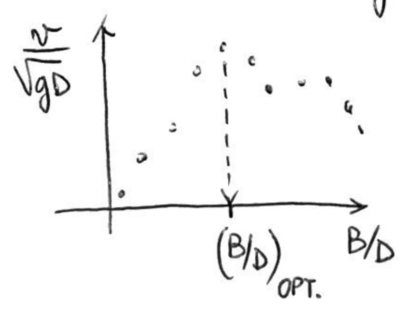
$$= \left(\frac{B}{D}\right)^c \left(\frac{h}{D}\right)^d \left(\frac{gD}{v^2}\right)^e$$

$$\Pi_1 \qquad \Pi_2 \qquad \Pi_3$$

$$\hat{F}(\Pi_1, \Pi_2, \Pi_3) = 0 \Rightarrow \Pi_3 = G(\Pi_1, \Pi_3) \Rightarrow v = \sqrt{gD}\,\hat{G}\left(\frac{B}{D}, \frac{h}{D}\right)$$

• Experiments may be carried to find for example how does the distance B play an influence on v ?

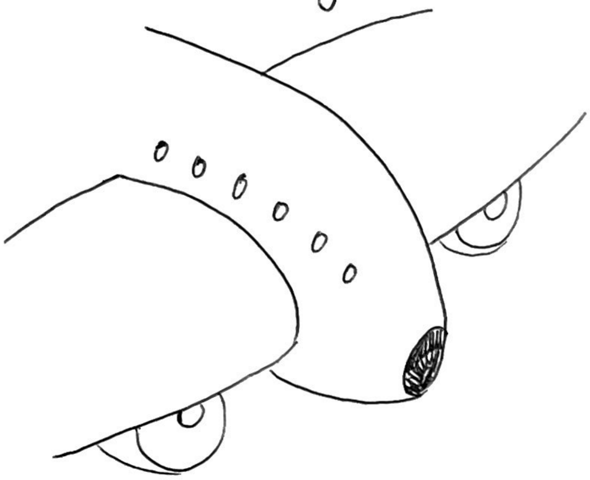
Neglect h/D & plot $v/\sqrt{gD}$ vs B/D



■ PHYSICAL SIMILARITY

Consider a PROTOTYPE of an airplane and its MODEL

Prototype



involved variables:

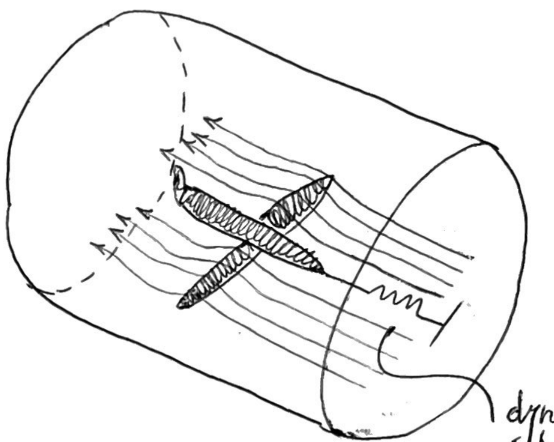
$$(\rho^P, \mu^P, F^P, H^P, v^P)$$

$$\Downarrow$$

Buckingham Th (see p. 8)

$$\Pi_1^P = G^P(\Pi_2^P) \qquad \begin{cases} \Pi_1^P = \dfrac{F^P}{H_P^2 \rho_P v_P^2} \\ \Pi_2^P = \dfrac{\mu_P}{H_P \rho_P v_P} \end{cases}$$

Model in wind tunnel



involved variables

$$(\rho^M, \mu^M, F^M, H^M, v^M)$$

$$\Downarrow$$

Buckingham Th

$$\Pi_1^M = G^M(\Pi_2^M) \qquad \begin{cases} \Pi_1^M = \dfrac{F^M}{\rho_M v_M^2 H_M^2} \\ \Pi_2^M = \dfrac{\mu_M}{H_M \rho_M v_M} \end{cases}$$

- Physical similarity exploits the fact that a physical phenomenon only depends on dimensionless variables; hence the effects taking place in the prototype and in the model are the same if

$$\begin{cases} G^P = G^M \\ \Pi_1^P = \Pi_1^M \\ \Pi_2^P = \Pi_2^M \\ \vdots \end{cases}$$

(until you saturate all the list of variables)

•) Example: A prototype of an airplane runs at $v_p = 800\ km/h$ in air (density ρ, dynamic viscosity μ). The engines exert a force F_P, completely balanced by air resistance in stationary conditions. Its size (length) is H_P.
A model of the same airplane is designed to fulfill physical similarity, it has length $H_M = \frac{1}{10} H_P$, it runs in air with same values as in the prototype.
A force $F_M = 15\ kN$ is measured on the model in a wind tunnel.
1) find the velocity v_M of the model that secures physical similarity.
2) What is the value of F_P ?

Solution: to ensure physical similarity impose:

$$\cdot\ \Pi_1^P = \Pi_1^M \Rightarrow \frac{F_P}{\rho v_P^2 H_P^2} = \frac{F_M}{\rho v_M^2 H_M^2} \Rightarrow F_P = F_M \cdot \left(\frac{v_P}{v_M}\right)^2 \left(\frac{H_P}{H_M}\right)^2 \quad (*)$$

$$\cdot\ \Pi_2^P = \Pi_2^M \Rightarrow \frac{\mu}{\rho v_P H_P} = \frac{\mu}{\rho v_M H_M} \Rightarrow v_M = \underset{800}{\underset{\downarrow}{v_P}} \left(\underset{10}{\frac{H_P}{H_M}}\right) = 8000\ km/h \checkmark$$

(EHM... a bit too much!) but ok it's only for theoret. purposes. Actually, the highest velocity in a wind tunnel is about 6500 km/h
So - if you want to reproduce the same physical effects with lower velocities, you need to change density & viscosity

Thus from (*) we get:

$$F_P = F_M \cdot \left(\frac{v_P}{v_M}\right)^2 \cdot \left(\frac{H_P}{H_M}\right)^2 \equiv F_M = 15\ kN.$$

— End of Dimensional Analysis —