NOISE AND BISTABILITY IN THE SQUARE ROOT MAP

Eoghan J. Staunton

eoghan.staunton@nuigalway.ie

Petri T. Piiroinen

NUI Galway
OÉ Gaillimh
IRISH RESEARCH COUNCIL
An Chomhairle um Thaighde in Éirinn

1 The Square Root Map

Recurrent mechanical real-world systems with impacts are often modelled using *impact oscillators*. Examples of such systems include rattling gears, moored boats impacting docks, Braille printers, percussive drilling and atomic force microscopes. Near low-velocity *grazing* impacts the dynamics of impact oscillators can be described by a one-dimensional map known as the square root map. We will consider the one-dimensional square root map with additive Gaussian white noise of amplitude Δ given by

$$x_{n+1} = S(x_n) = \begin{cases} S_L(x_n) = \mu + bx_n + \xi_n, & x_n < 0, \\ S_R(x_n) = \mu - a\sqrt{x_n} + \xi_n, & x_n \geq 0, \end{cases} \quad (1)$$

where $\xi_n \sim N(0, \Delta^2)$, $S_L(x)$ is the linear part of the map applied on the left-hand side when $x < 0$, and $S_R(x)$ is the square root part applied on the right when $x \geq 0$.

2 The Period-Adding Cascade

Here we will assume that $a > 0$ and $0 < b < \frac{1}{4}$. In this case the deterministic square root map ($\Delta = 0$) undergoes a period-adding cascade with intervals of bistability as the bifurcation parameter μ is decreased. This structure can be clearly seen in Figure 1.

We see that there are values of $\mu > 0$ for which a stable periodic orbit of period m exists for each $m = 2, 3, \ldots$, and other values of $\mu > 0$ such that there are two stable periodic orbits coexisting, one of period m and the other of period $m+1$.

Note that all attractors have symbolic codes of the form $(RL^n)^\infty$ meaning that they have exactly one iterate on the right (R) and the remaining n iterates are on the left (L).

3 Adding Noise

Our interest is in the qualitative behaviour of the square root map in the presence of additive white noise. In particular we focus on the effect of noise of varying amplitudes on systems with values of in, or close to, the intervals of bistability, for which stable periodic orbits of period m and period $m+1$ coexist. In these regions complicated deterministic structures interact with noise to produce interesting dynamics.

4 Why Consider Noise?

Traditionally mathematicians have used smooth deterministic models to model the real world. These models present a simplified view of the world where the evolution of systems exhibits no interruptions such as impacts, switches, or jumps and there is no uncertainty (or noise) present. However, independently, both non-smoothness and noise have been shown to drive significant changes in the behaviour of a model. It is therefore important to investigate and understand how the inclusion of both non-smoothness and noise can affect the behaviour of a model.

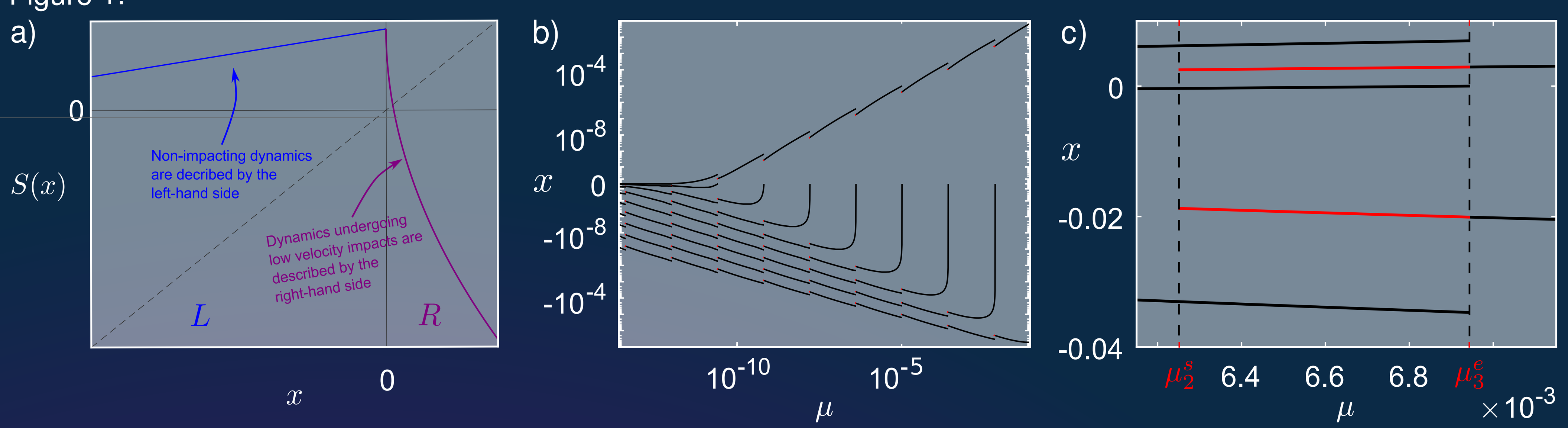
5 Noisy Dynamics

We focus our investigation on the effect of noise on the system for values of μ in, or close to, the interval of coexistence for period-2 and 3 attractors shown in Figure 1c).

We find that adding noise of low amplitude to the system initially causes the interval of coexistence to effectively shrink. However, increasing the noise amplitude we find this trend reverses, in fact we even begin to see persistent period-2 (RL) behaviour in the region where the period-2 orbit is unstable in the deterministic square root map. This can be clearly seen in Sector 4.

The schematic in Figure 2 shows how the relationship between noise amplitude and the periodic behaviour varies on the interval of bistability. We see that the relationships are non-monotonic and highly dependent on the value of μ. These relationships are examined in detail in [1]. Here we will concentrate on how noise effectively stabilises period-2 behaviour in a region where it is unstable in the deterministic system.

Figure 1.



a) The deterministic square root map. b) Period-adding bifurcation cascade in the deterministic square root map with $b < \frac{1}{4}$. c) A close-up view of the interval of bistability where period-2 and period-3 attractors coexist.

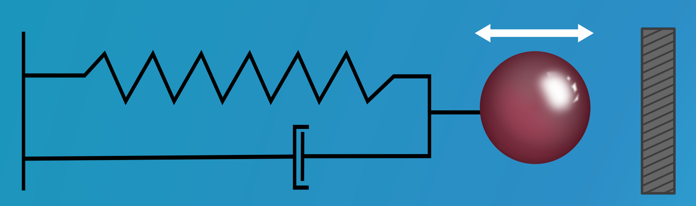
6 The Transition Mechanism

Perhaps the most interesting phenomenon that we have observed is the potential for repeated intervals of persistent period-2 dynamics in a noisy system with μ such that the period-2 orbit is unstable in the corresponding deterministic system. We observe that the noise-induced transitions from period-3 to period-2 behaviour in regions where period-2 behaviour is unstable tend to take the following symbolic form

$$RLLRLL \ldots RLLRLRRLRL \ldots RLRL. \quad (2)$$

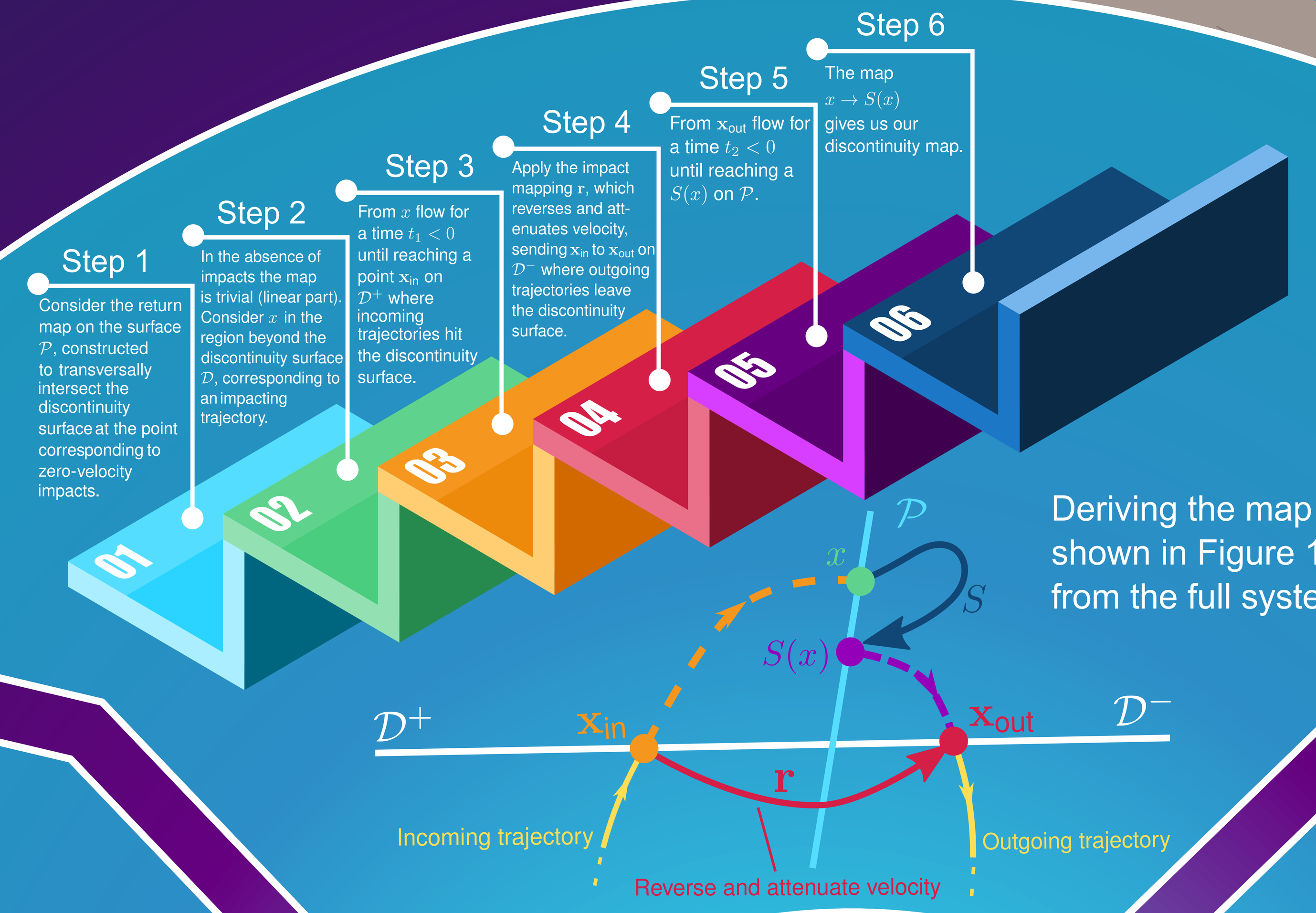
The significant feature of the symbolic representation of the transition above is the repeated R, corresponding to repeated iteration on the right-hand side of the square root map, i.e. repeated low-velocity impacts in the physical system. These repeated low-velocity impacts allow the dynamics to be pushed into the region of phase space with slow dynamics, in the vicinity of the unstable period-2 orbit of the deterministic system. The system can then take a significant number of iterates to transition back to behaviour. In fact, once close to the unstable orbit noise can have a stabilising effect.

1



A forced impact oscillator.

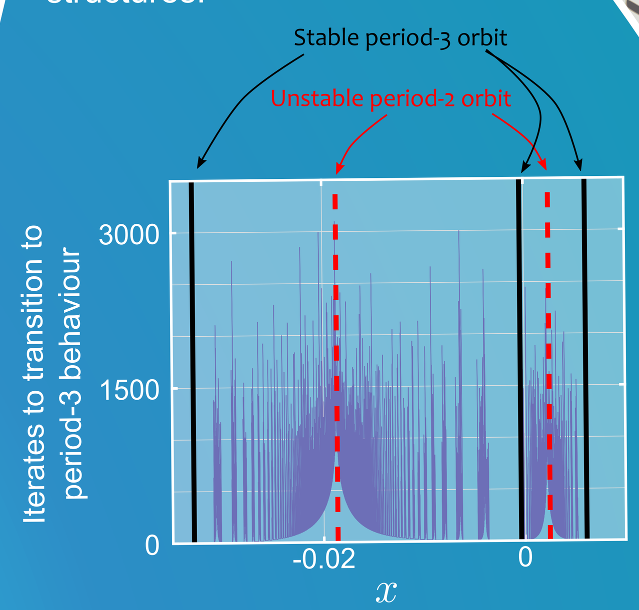
2



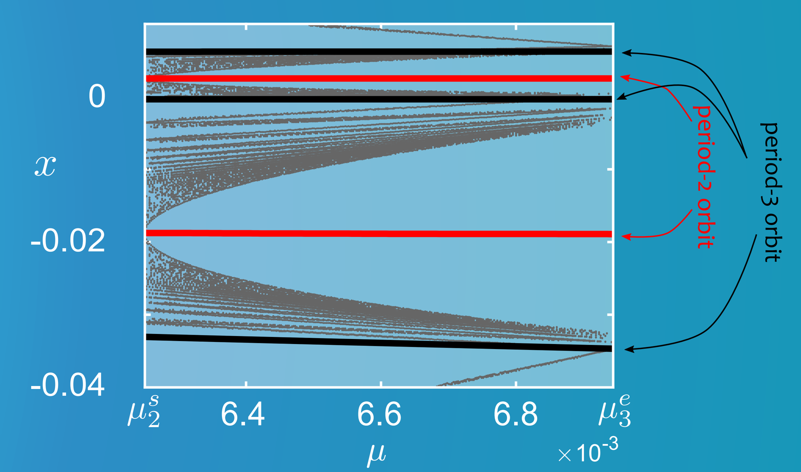
Deriving the map shown in Figure 1a) from the full system.

3

The nonsmooth nature of the square root map creates complicated deterministic structures.

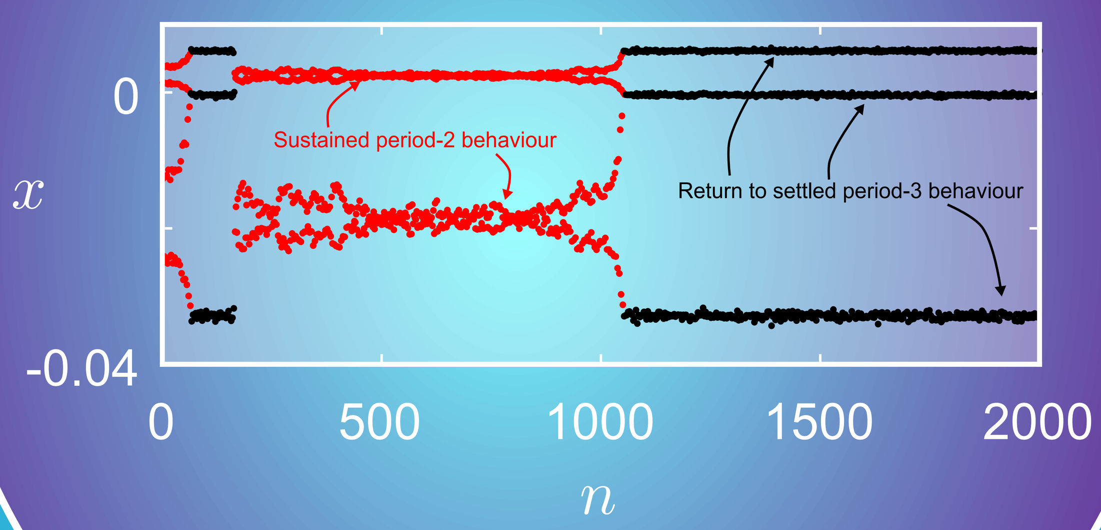


For values of μ close to the interval of bistability, where the period-2 orbit is unstable, we see that the relationship between the time taken to transition to period-3 behaviour and our initial condition is very complicated.



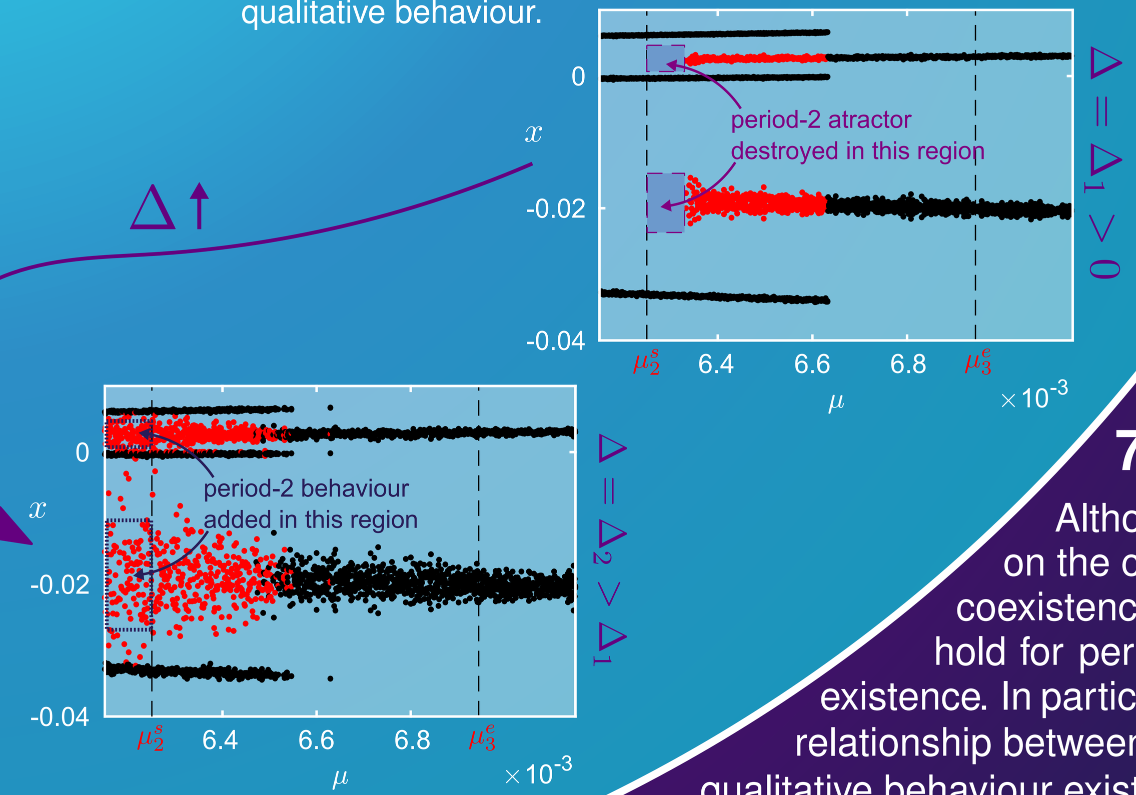
On the interval of bistability where both the period-2 and the period-3 attractors are stable the basins of attraction have a fine riddled structure. Here we show the basin boundaries in grey.

Sample Transition



4

Adding noise of small but increasing amplitude, $|\Delta| \ll 1$, leads to a non-monotonic response in qualitative behaviour.



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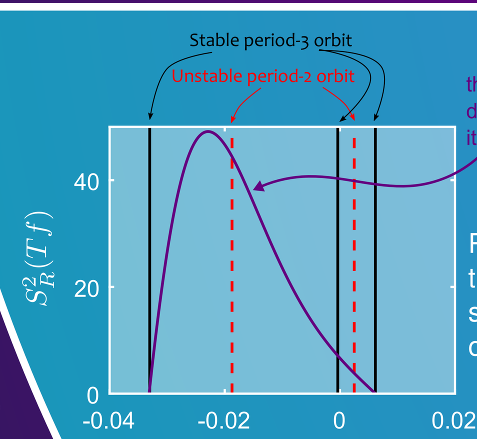
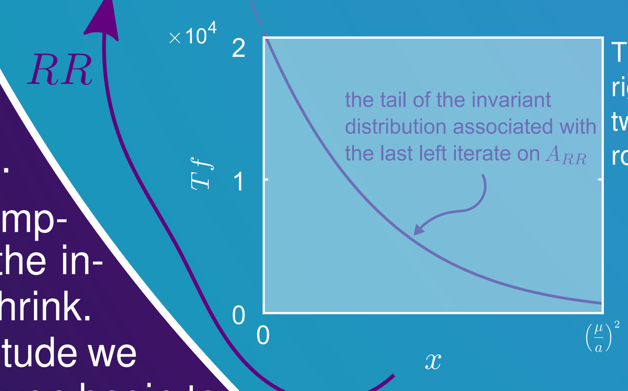
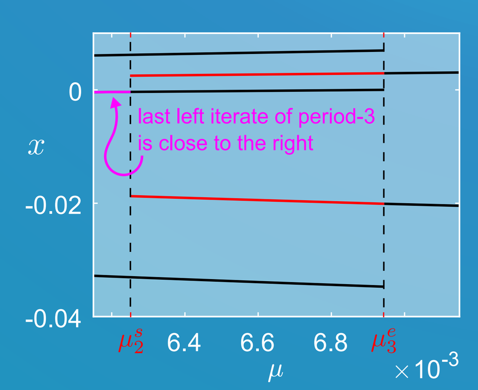


Repeated low velocity impacts (or iterations on the right) concentrate trajectories around the unstable period-2 orbit, causing transitions like the one seen in the centre of the poster.



The interval $A_{RR} = (0, (\mu/a)^2)$ located just the right of 0 is the set of points that will be iterated twice consecutively on the right by the square root map.



The last left iterate of the deterministic period-3 orbit is close to zero. It is not hard to imagine that this iterate could be pushed onto the right by low amplitude additive noise.

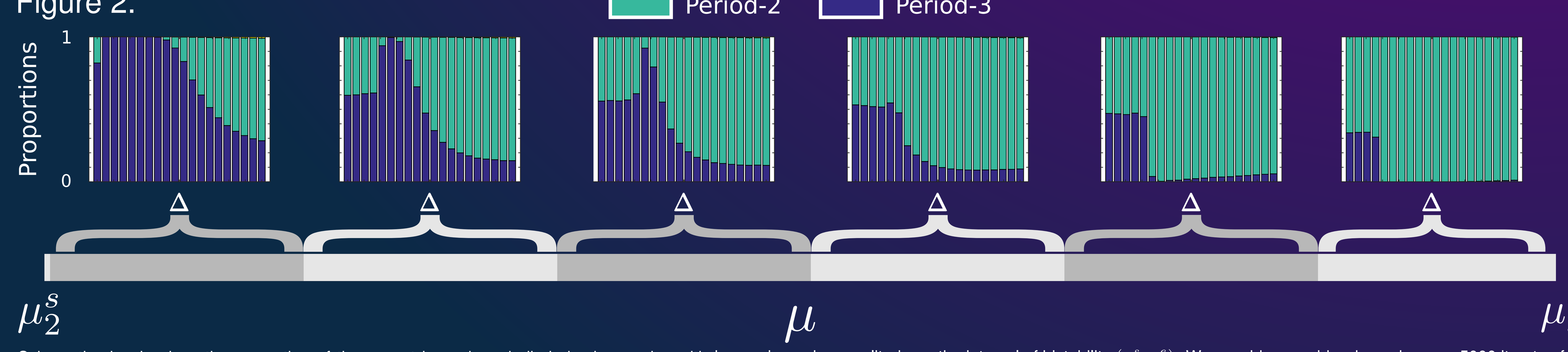
7 Generalising

Although we have focused on the case of period-2 and 3 coexistence here, similar results hold for period-m and $m+1$ coexistence. In particular, a non-monotonic relationship between noise amplitude and qualitative behaviour exists. We find that noise-induced transitions from period-m to period-$m+1$ behaviour in the region where the period-m orbit is unstable take the following form

$$RL^mRL^m \ldots RL^m \underline{RL^{m-1}RL^{k-2}RL^{m-1}}RL^{m-1} \ldots\ldots RL^{m-1} \quad (3)$$

where $k \in \{2, 3, \ldots, m\}$. The most significant feature of this transition is the sequence $RL^{k-2}R$ for $k \in \{2, 3, \ldots, m\}$, again corresponding to the repetition of low-velocity impacts in quick succession, forcing the dynamics into the region of phase space close to the unstable period-m orbit.

Figure 2.



Schematic showing how the proportion of time spent in each periodic behaviour varies with increasing noise amplitude on the interval of bistability (μ_2^s, μ_3^e). We consider consider dynamics over 5000 iterates for 1000 different orbits with linearly spaced initial conditions and the first column of each bar chart shows the proportions in the deterministic case ($\Delta = 0$).

References

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