Crystallography and Art

Bernd Souvignier
Radboud University Nijmegen

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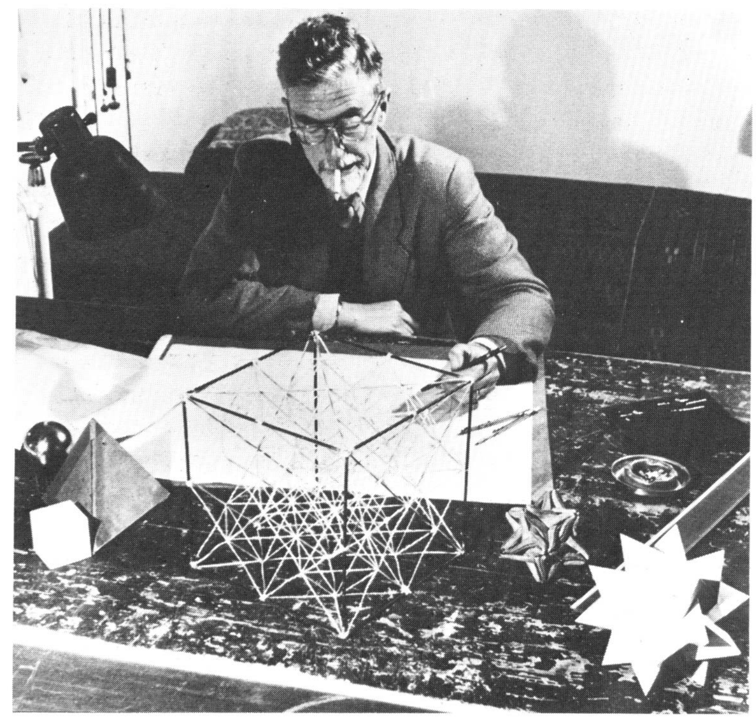
Overview

- M.C. Escher
- Penrose patterns
- Quasicrystals
- Islamic Art

M.C. Escher

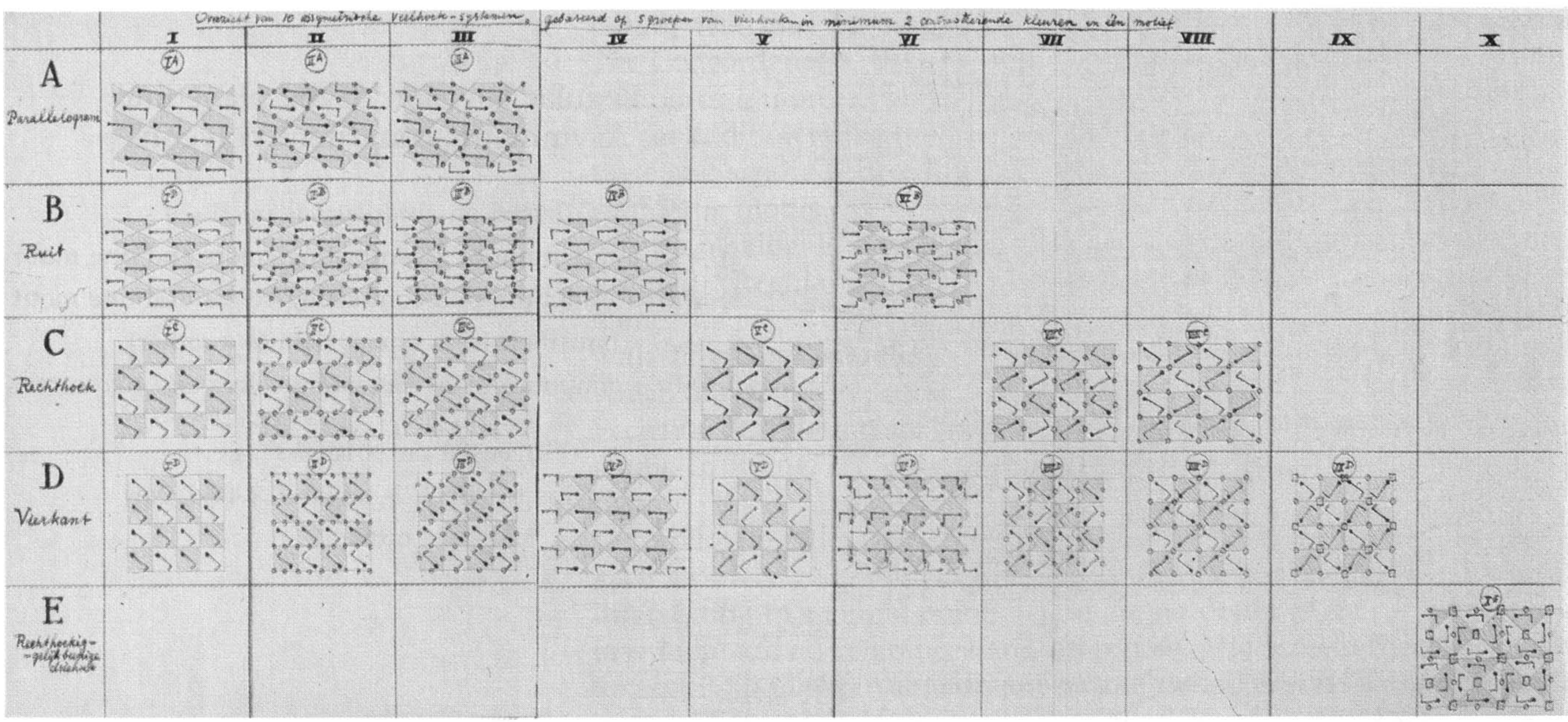
Dutch artist, 1898 - 1972

Famous e.g. for: regular plane tilings, impossible figures



Theoretical work

Escher developed his own classification of regular plane tilings, partially coinciding with the crystallographic approach (e.g. based on lattice types), partially contrasting it.



Relation to the crystallographic community

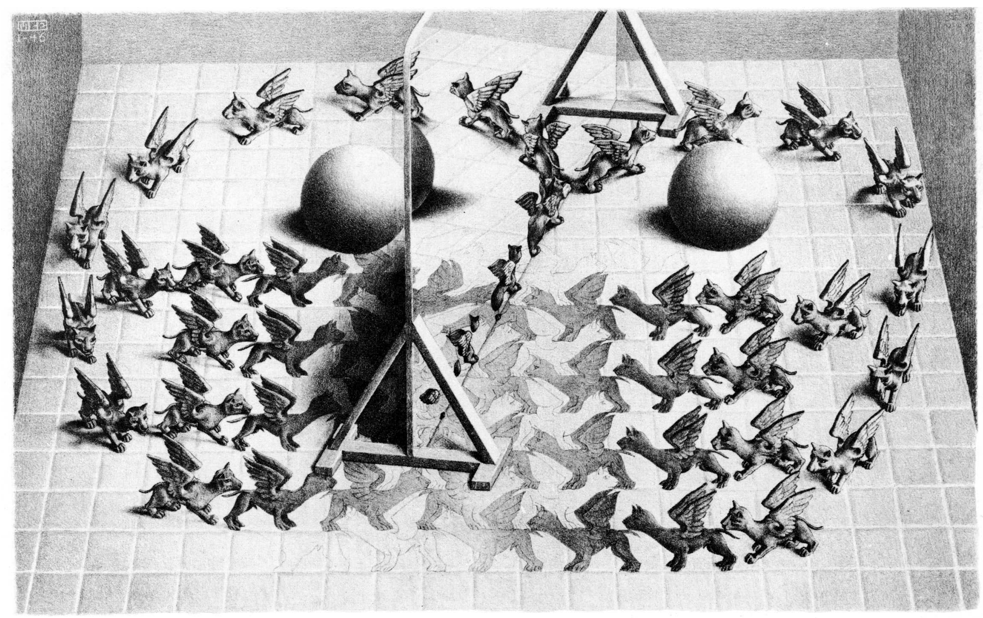
Escher was invited to give a plenary talk at the International Congress of Crystallography in Cambridge in 1960.

In his abstract, he states:

From the beginning of my investigations, the use of contrasting colours or shades was both self-evident and necessary in order to distinguish visually between neighbouring figures. I was therefore surprised to learn that the notion of antisymmetry has only recently been introduced into and accepted by crystallography. My own applications are unthinkable without the use of colour contrast.

Illusion of space

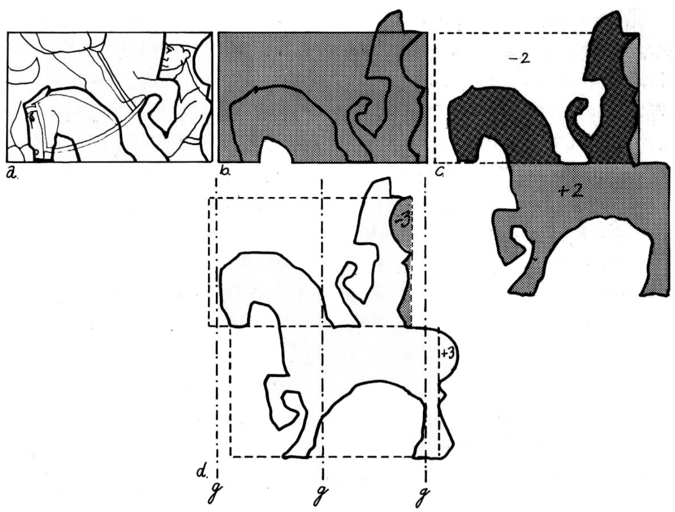
While creating regular plane tilings, Escher enjoyed playing with the illusion of representing a 3-dimensional reality.

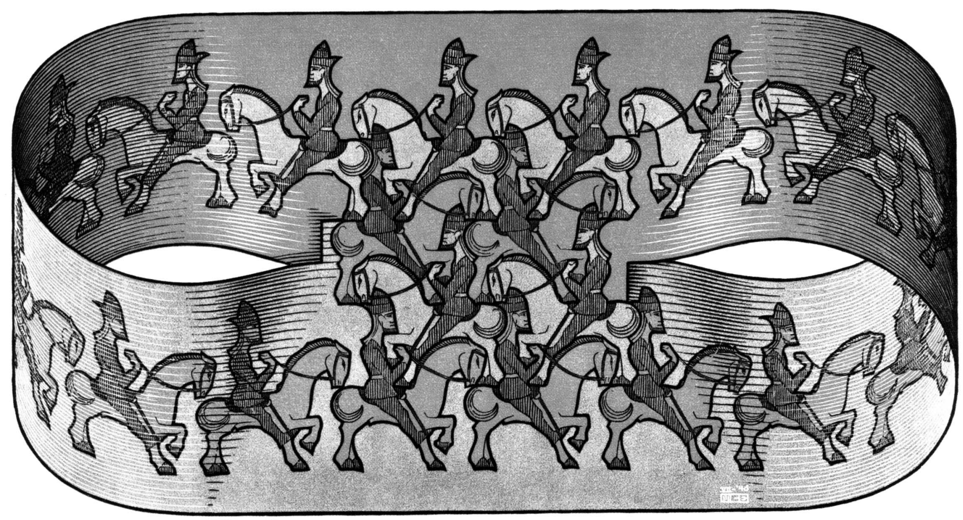


Lattice-equal subgroup of index 2.

Construction of a plane filling tile

Often, the construction of the figures is highly ingenious.

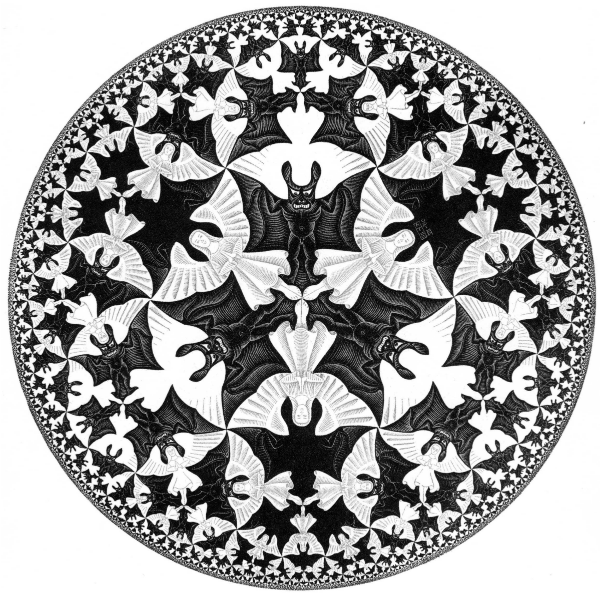


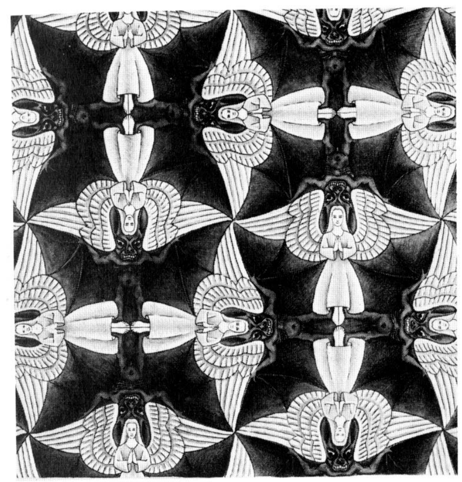


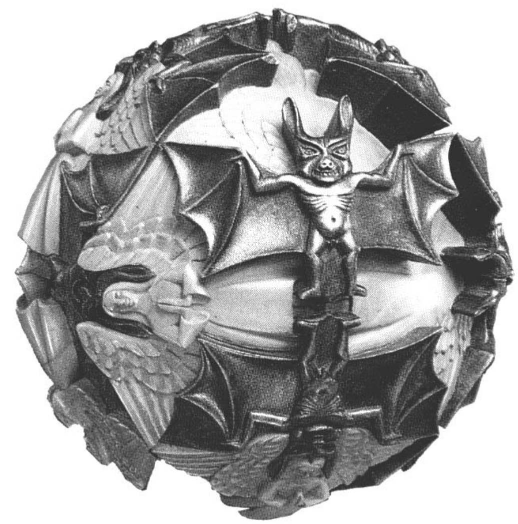
The plane groups of Escher's patterns almost never allow proper reflections, since that would require the figures to have straight edges.

Other geometries

Inspired by an illustration in a paper of H.S.M. Coxeter, giving a regular tiling of a hyperbolic plane, Escher also worked in planes with curvature $\neq 0$.



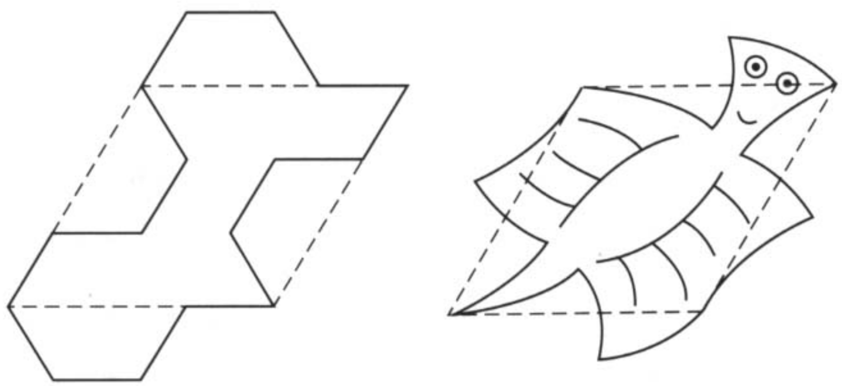


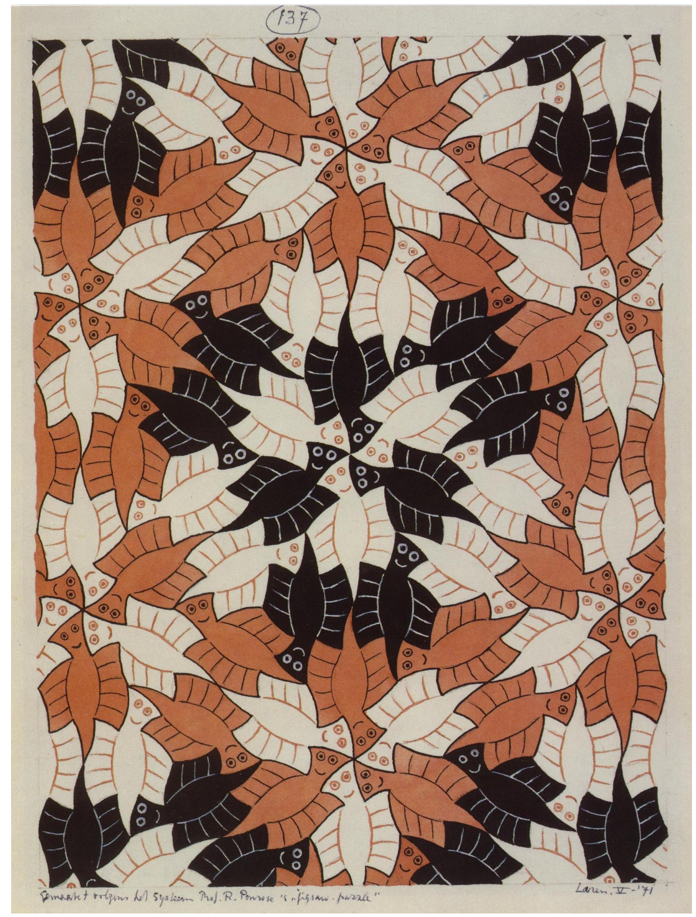


Plane tilings with triangles of angle sum $\frac{5}{6}\pi$ (hyperbolic), π (Euclidean) and $\frac{7}{6}\pi$ (elliptic). Note that the same motif is used in all three cases.

Ghosts

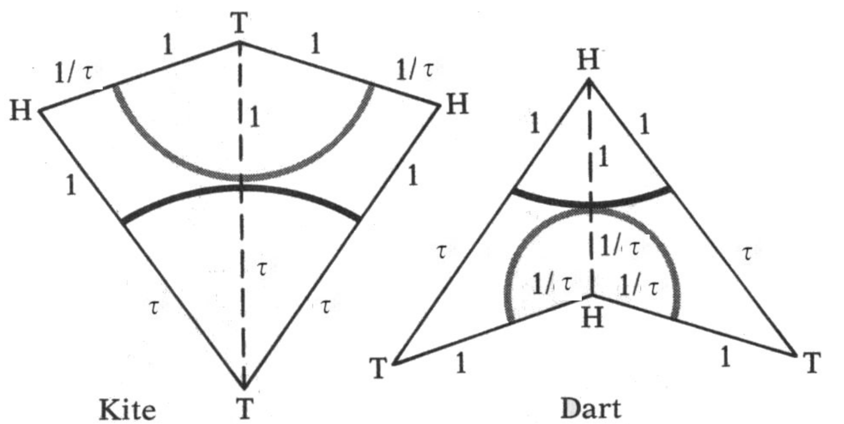
In 1962, R. Penrose challenged Escher with a set of congruent jigsaw pieces which would tile the plane in a unique way. Based on these tiles, Escher designed his last print *Ghosts* (1971).

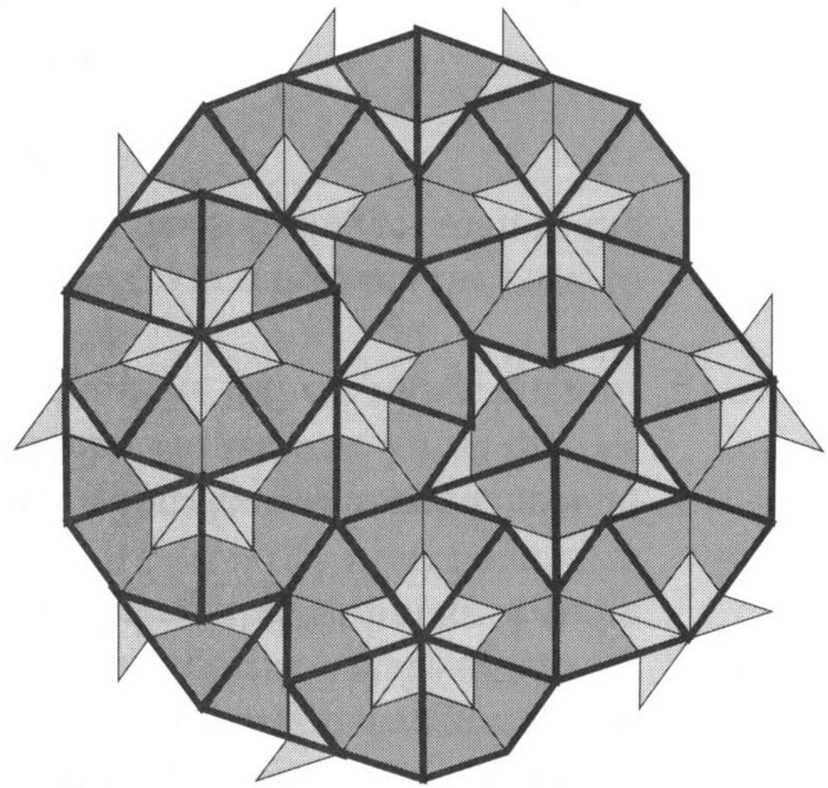




Penrose tiles (kites and darts)

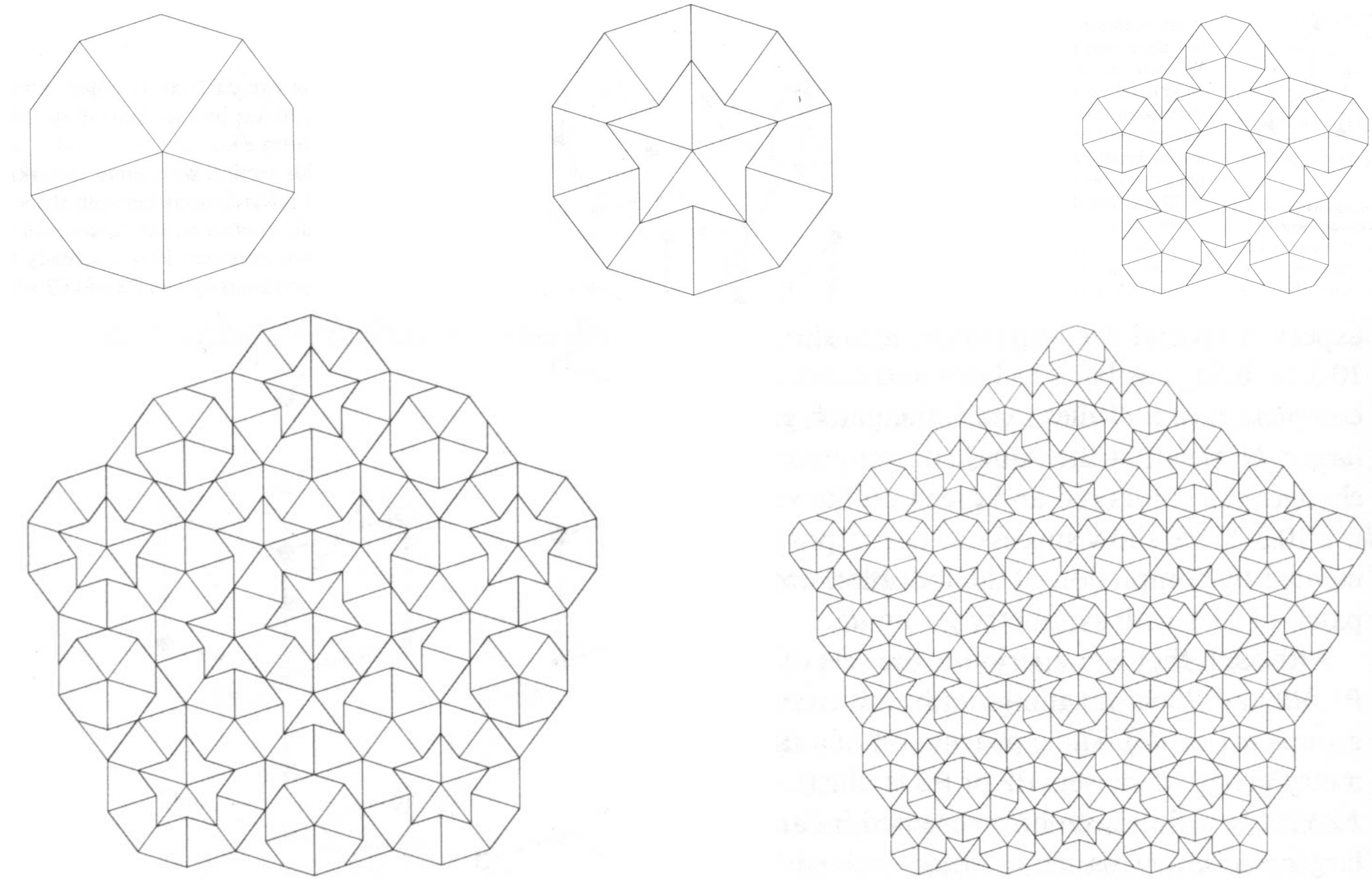
Developed out of the subdivision of a pentagon into smaller pentagons, filling the gaps with other tiles which allow iteration of the subdivision.





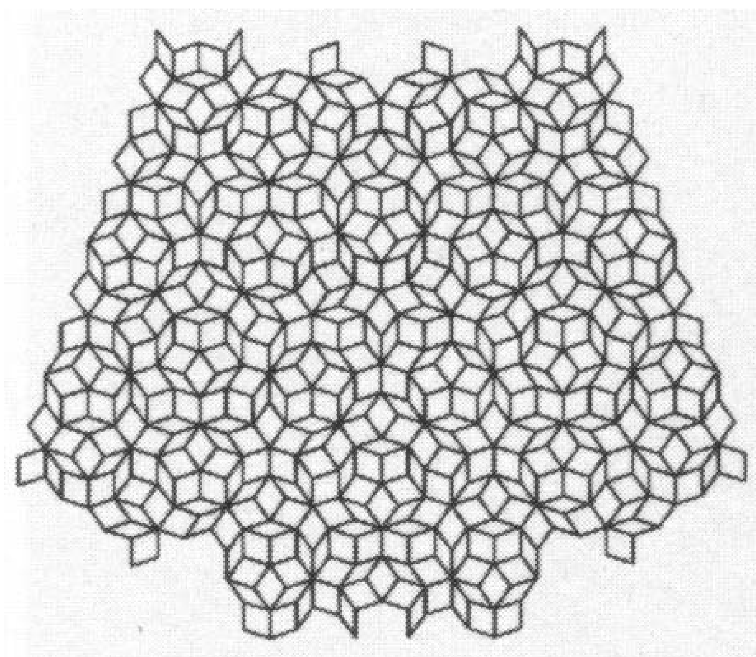
Self-similarity with scaling factor $\tau = \frac{1+\sqrt{5}}{2} \approx 1.618$

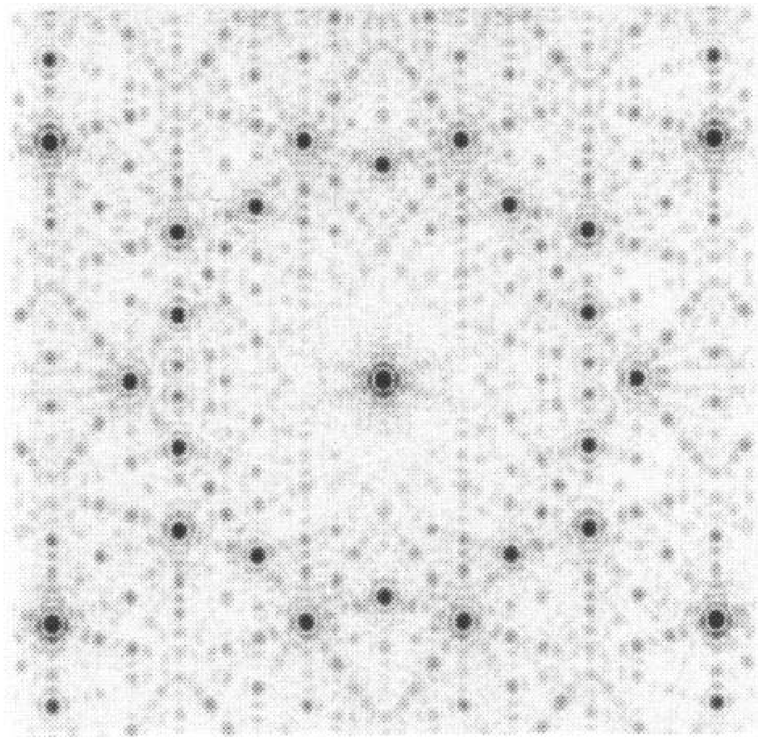
Non-periodic tiling via subdivision and inflation



Diffraction from Penrose tilings

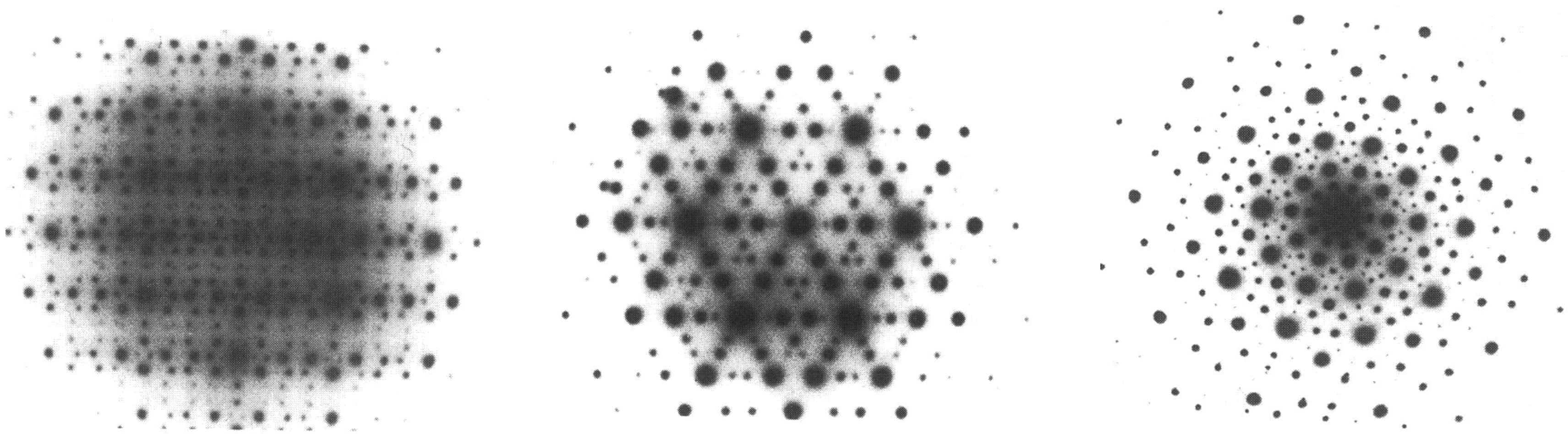
Despite its non-periodicity, the diffration pattern of a Penrose tiling (with the vertices of the tiles as diffraction gratings) displays sharp Bragg peaks.
The decagonal symmetry of the diffraction pattern is incompatible with a 2-dimensional lattice.





Quasicrystals

In 1984, Shechtman published an article with diffraction patterns of an $Al\,Mn$-alloy, displaying 2-fold, 3-fold and 5-fold rotational axes and in fact icosahedral symmetry.



Conflict: Sharp Bragg peaks $\Rightarrow$ crystal; 5-fold rotation axis $\Rightarrow$ non-periodic $\Big\} \Rightarrow$ quasicrystal

Cut-and-projection method

Quasicrystals can be described as certain projections of periodic structures in higher-dimensional spaces to the real space (usually $\mathbb{R}^3$ or $\mathbb{R}^2$).

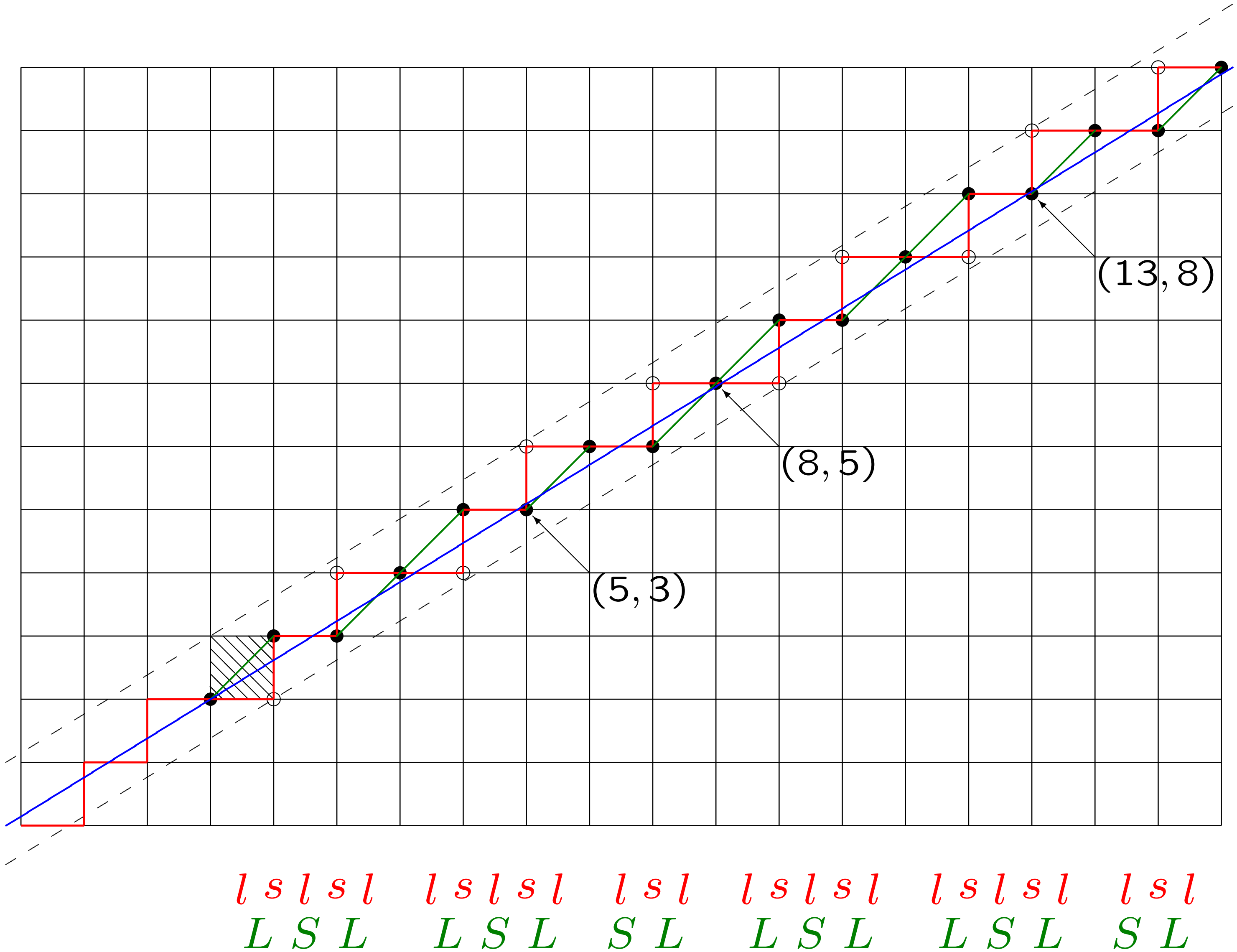
Let $\mathbb{R}^n = V_P \perp V_I$ be an orthogonal decomposition, where V_P is called the physical or external space and V_I is called the internal space.

A (periodic) structure $\mathcal{S} \subset \mathbb{R}^n$ is projected to the physical space V_P by mapping $v = v_p + v_i$ to v_p if v_i lies in some compact subset $\mathcal{C}$ of $\mathbb{R}^n$, i.e. if v is close to V_P.

A common choice for the window $\mathcal{C}$ is the Voronoï cell of $\mathcal{S}$.

In order to preserve certain symmetry elements, the external space V_P is typically chosen as an $\mathbb{R}$-irreducible submodule of a subgroup of $\Pi(G)$ (where G is the space group of $\mathcal{S}$).

Fibonacci quasicrystal



$$x = \tau y$$

$$\tau := \frac{1+\sqrt{5}}{2}$$

Self-similarity

A scaling of the window $\mathcal{S}$ in the Fibonacci quasicrystal by τ induces the self-similarity transformation

$$S \to L, \quad L \to S\ L.$$

for the Fibonacci quasicrystal.

The self-similarity property holds generally for the cut-and-project method.

Together with the fact, that a periodic structure in the higher-dimensional space is projected, this explains the sharp Bragg peaks in the diffraction pattern.

The Penrose pattern is cubic

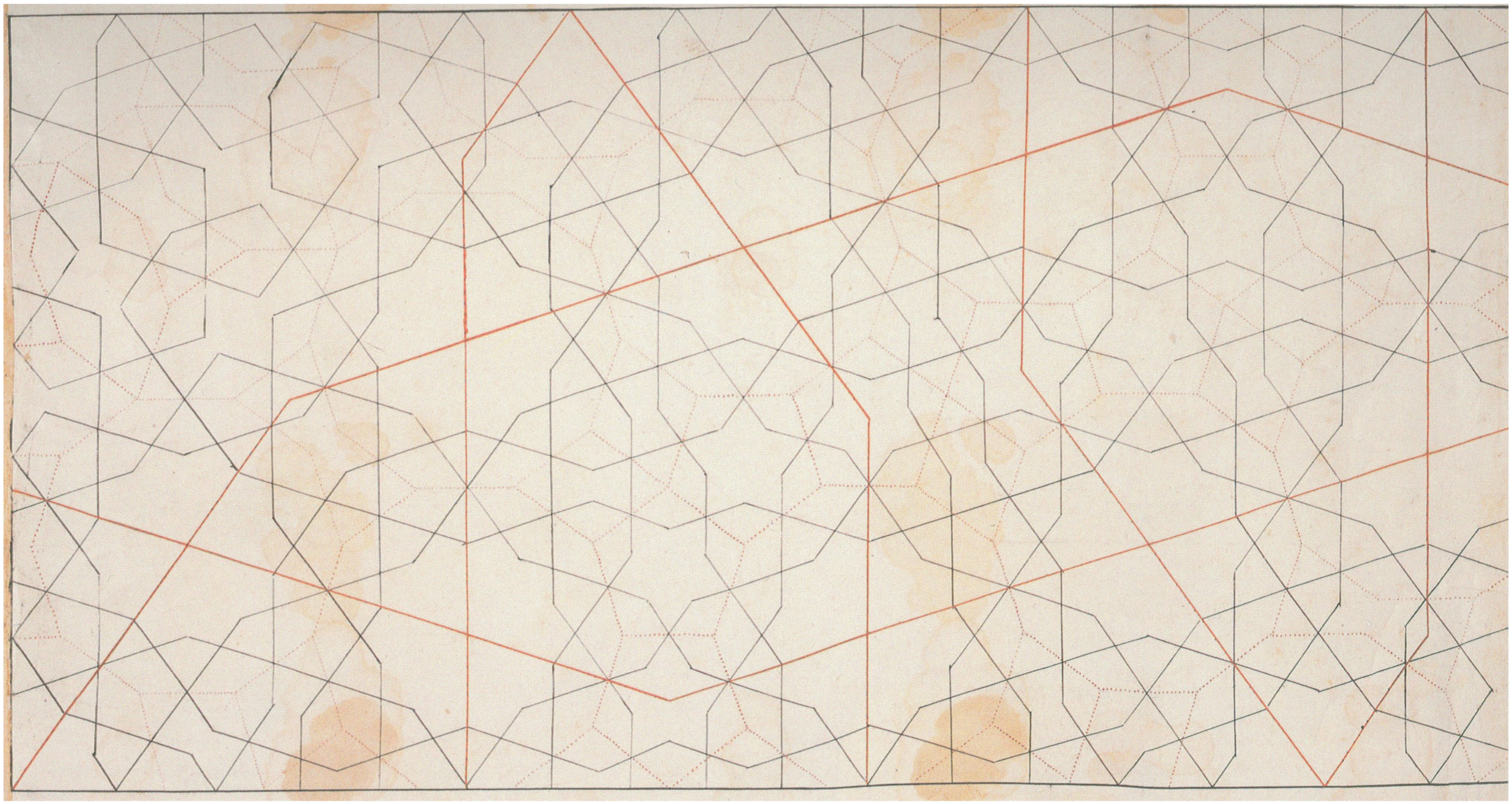
The Penrose pattern (with thick and thin rhombs) can be obtained by the cut-and-project method from the 5-dimensional cubic lattice $\mathbb{Z}^5$.

This lattice admits a 5-fold rotation permuting the vertices adjacent to one vertex. One such axis is along the vector $v = (1, 1, 1, 1, 1)^{tr}$.

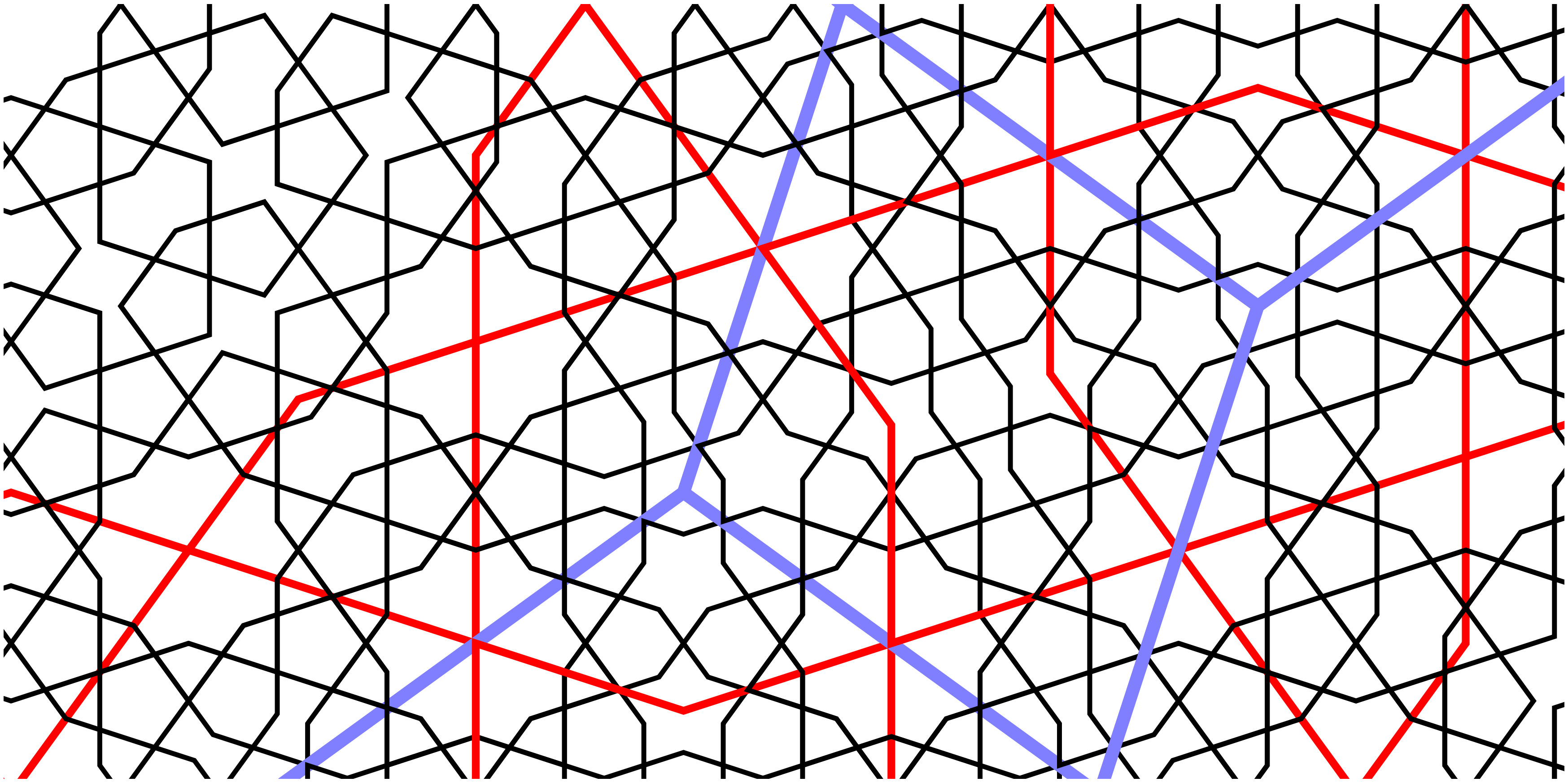
The 4-dimensional space $v^\perp$ splits into two 2-dimensional real subspaces invariant under the 5-fold rotation. One of these is chosen as physical space.

Thus, the Penrose pattern is designed to have 5-fold rotation symmetry (even dihedral), whereas the symmetry elements of $\mathbb{Z}^5$ that do not fix the physical space are lost (in particular all translations).

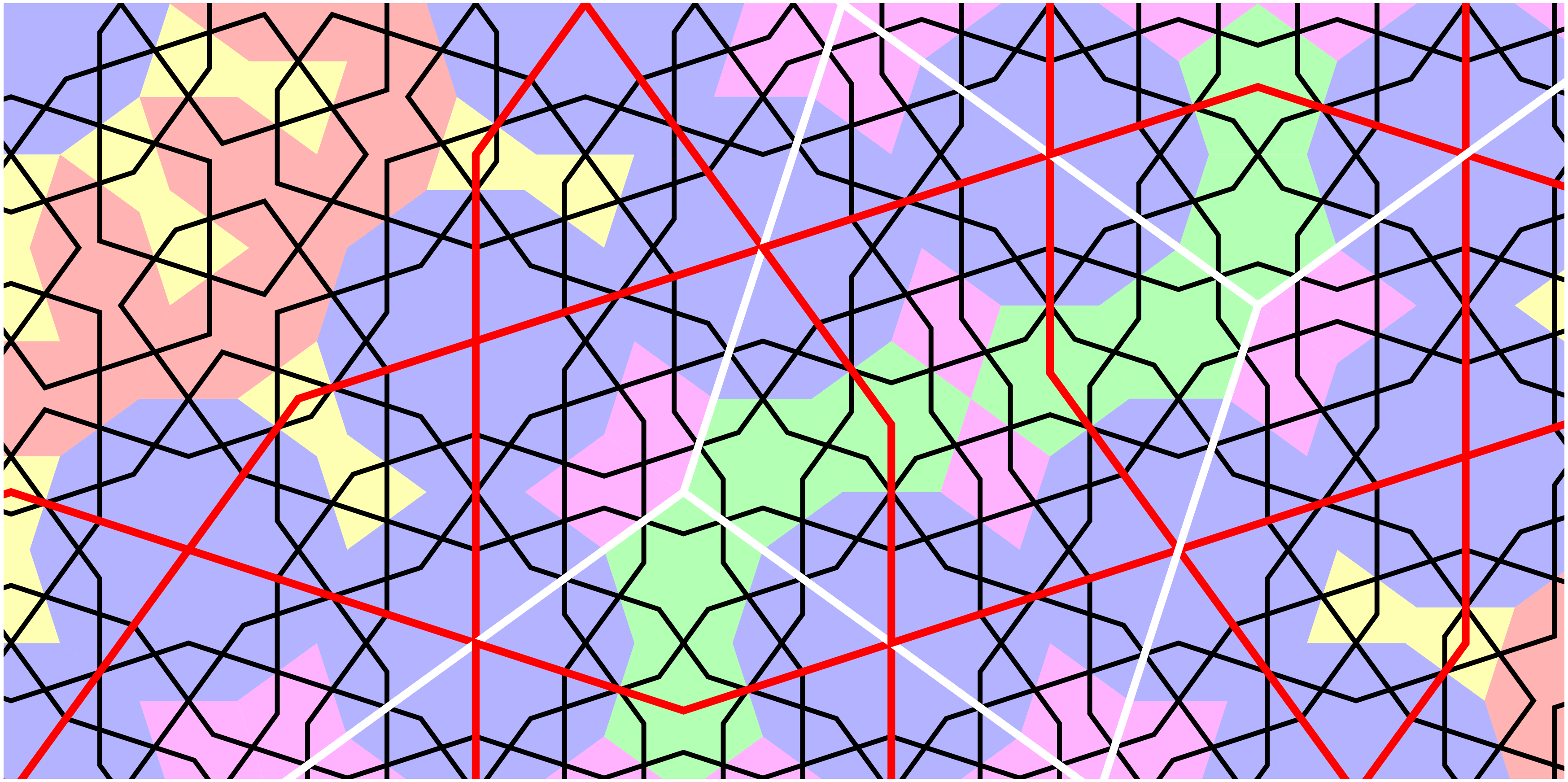
Pattern no. 28 from the Topkapi scroll (~ 1500)



Girih (line) pattern

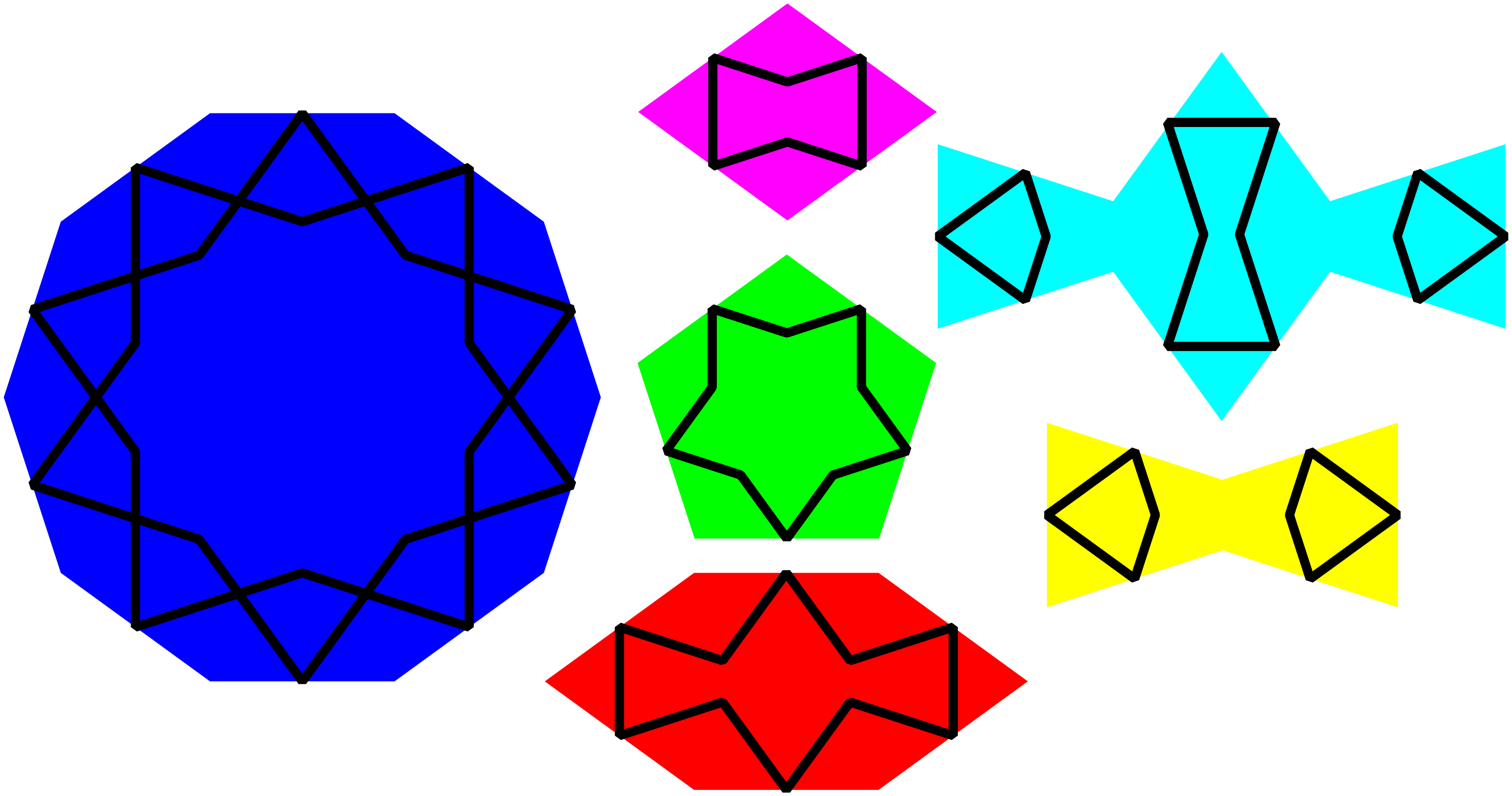


Girih pattern with underlying tiles



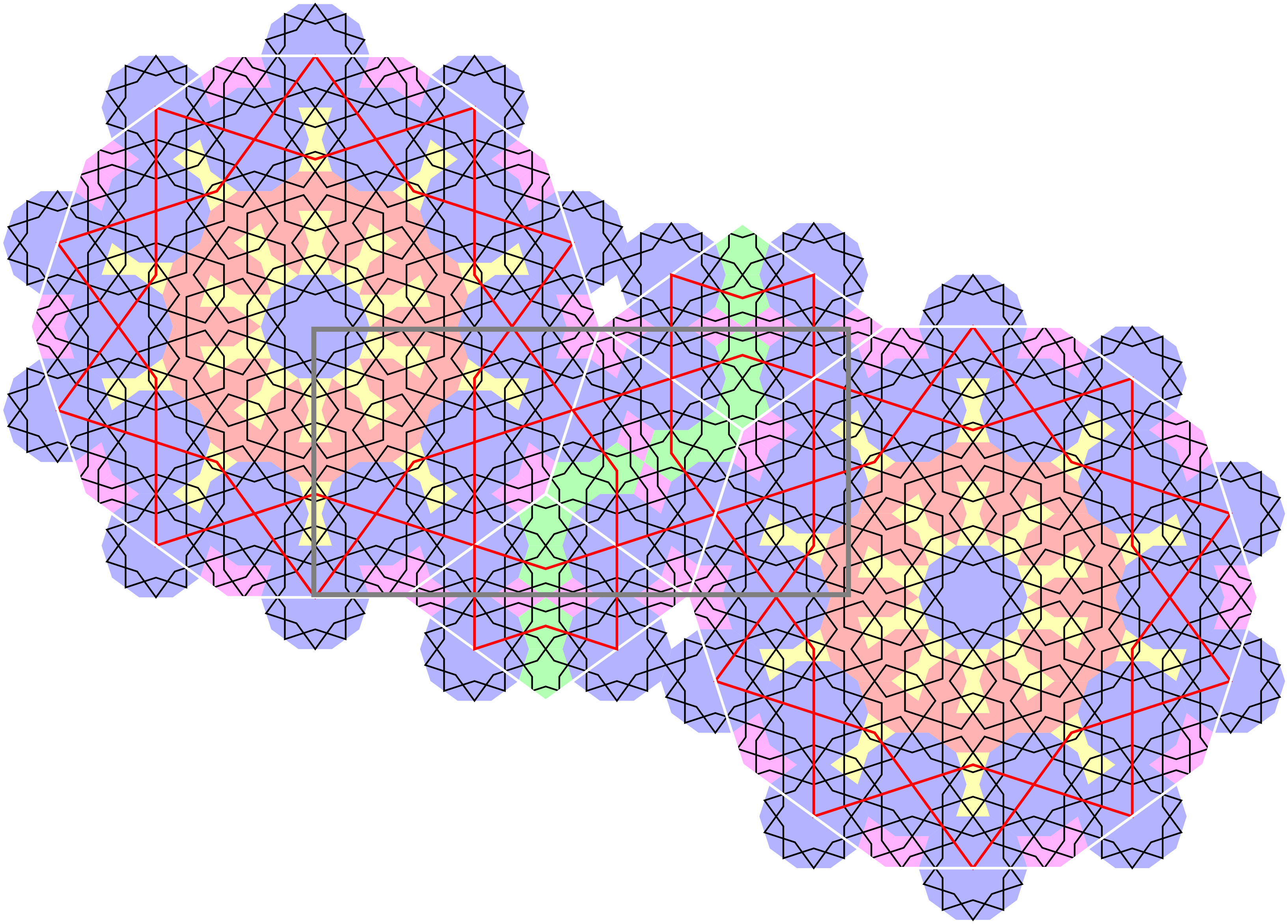
The tiles are indicated by red-dotted lines in the Topkapi scroll

Girih tiles from pattern no. 28

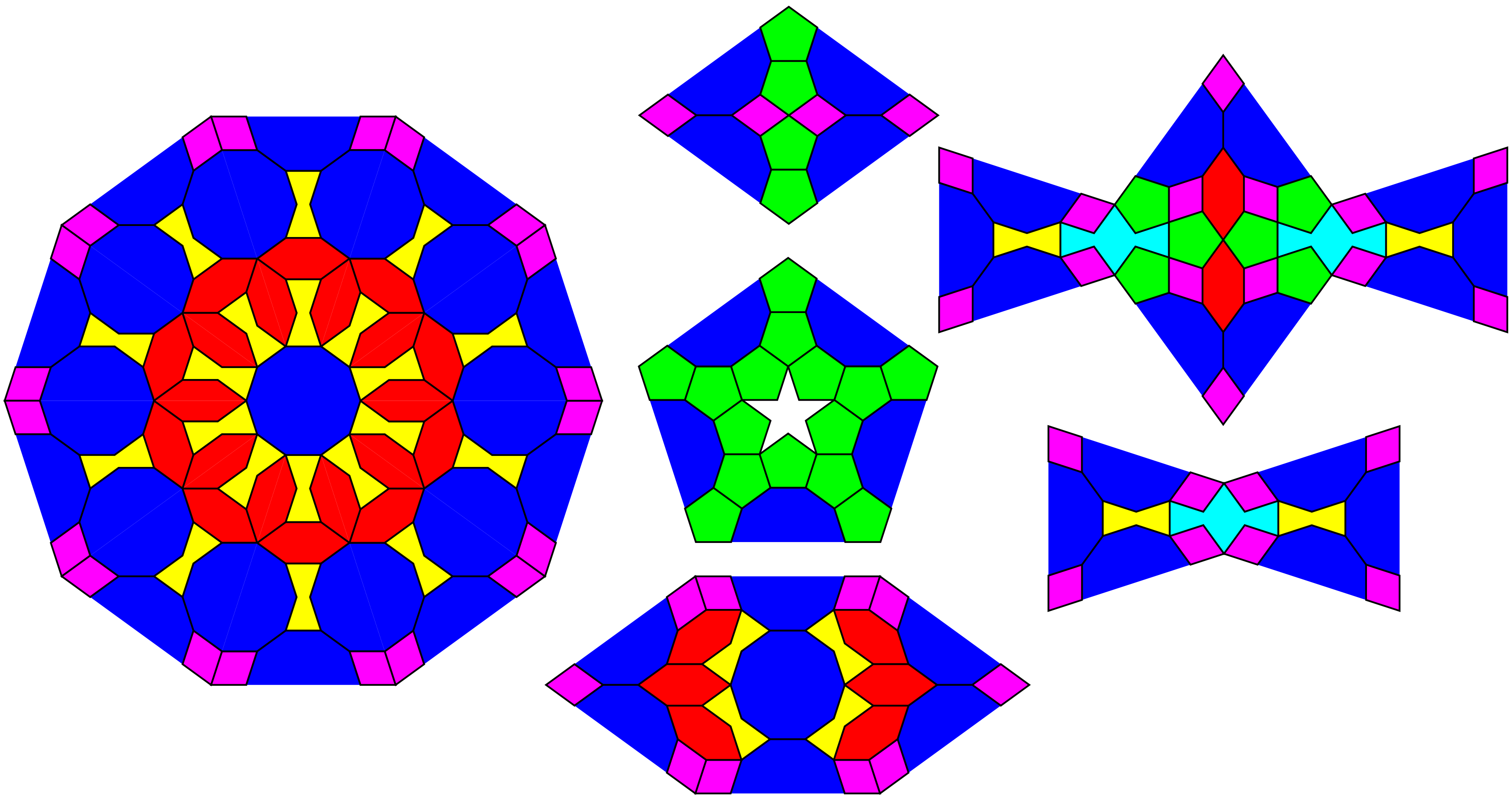


except for upper right ‘bat’

Large-scale tiles

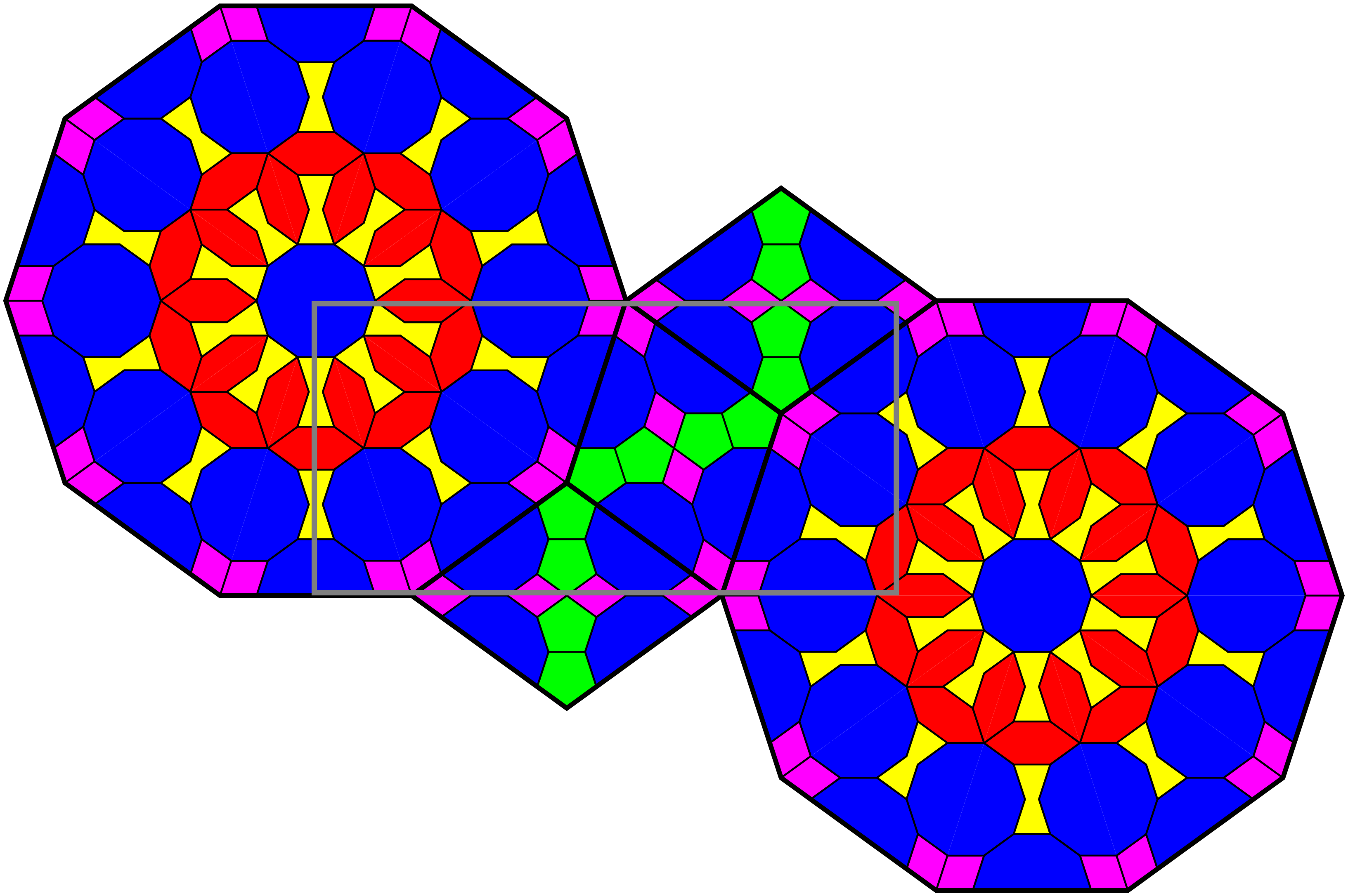


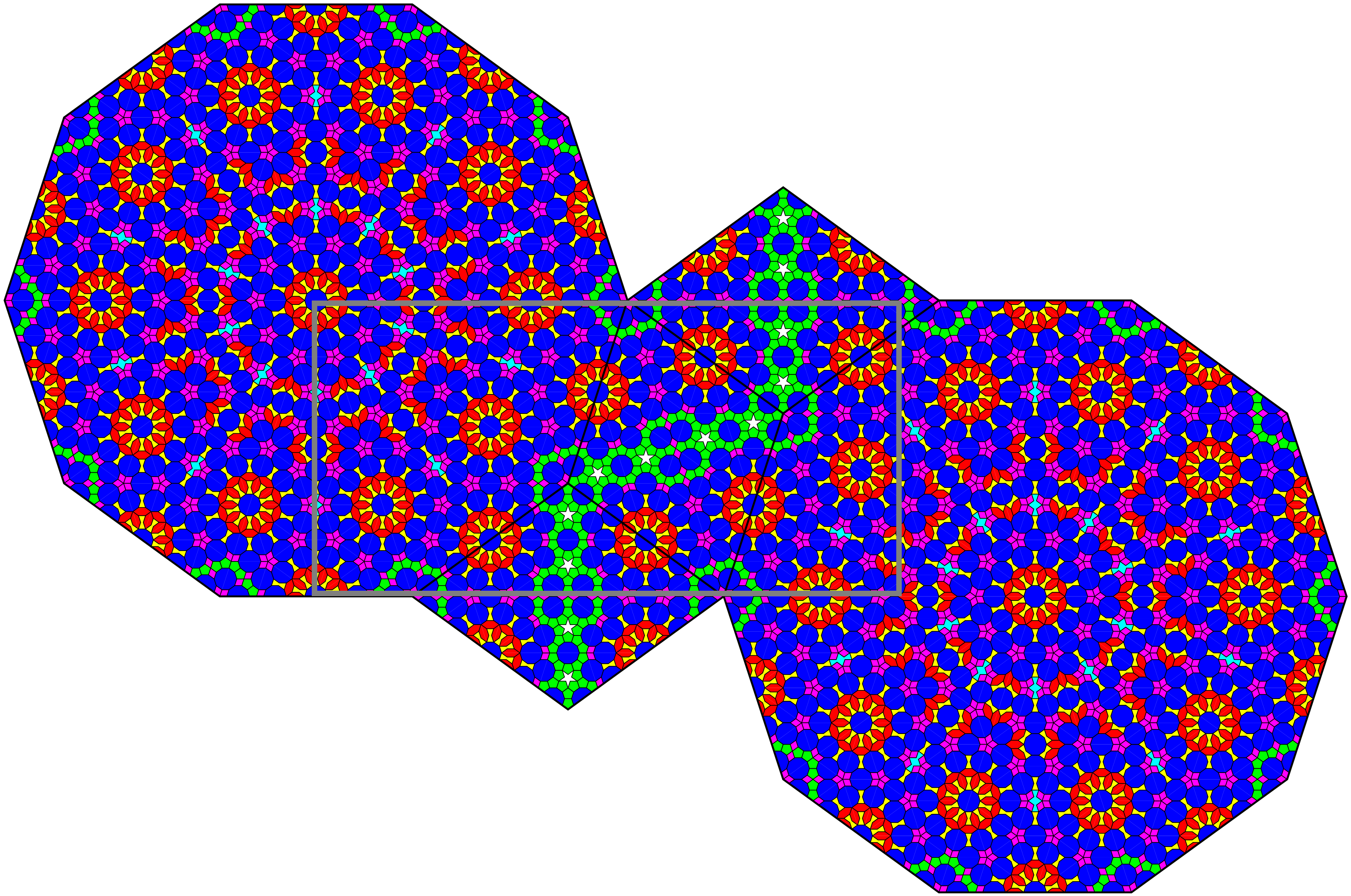
Subdivision of girih tiles



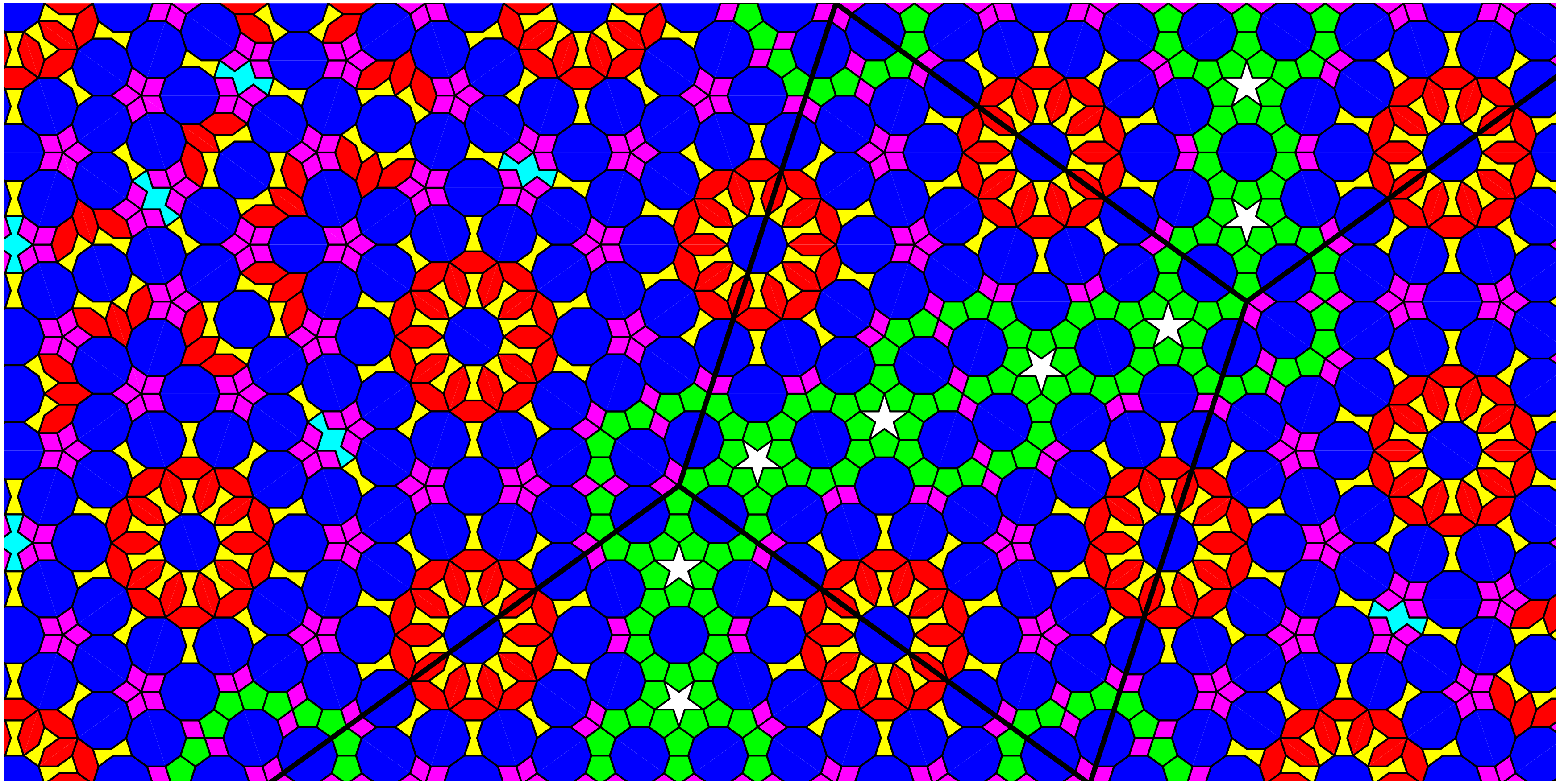
Scaling factor $2\tau + 2 \approx 5.236$

Further subdivision of pattern no. 28



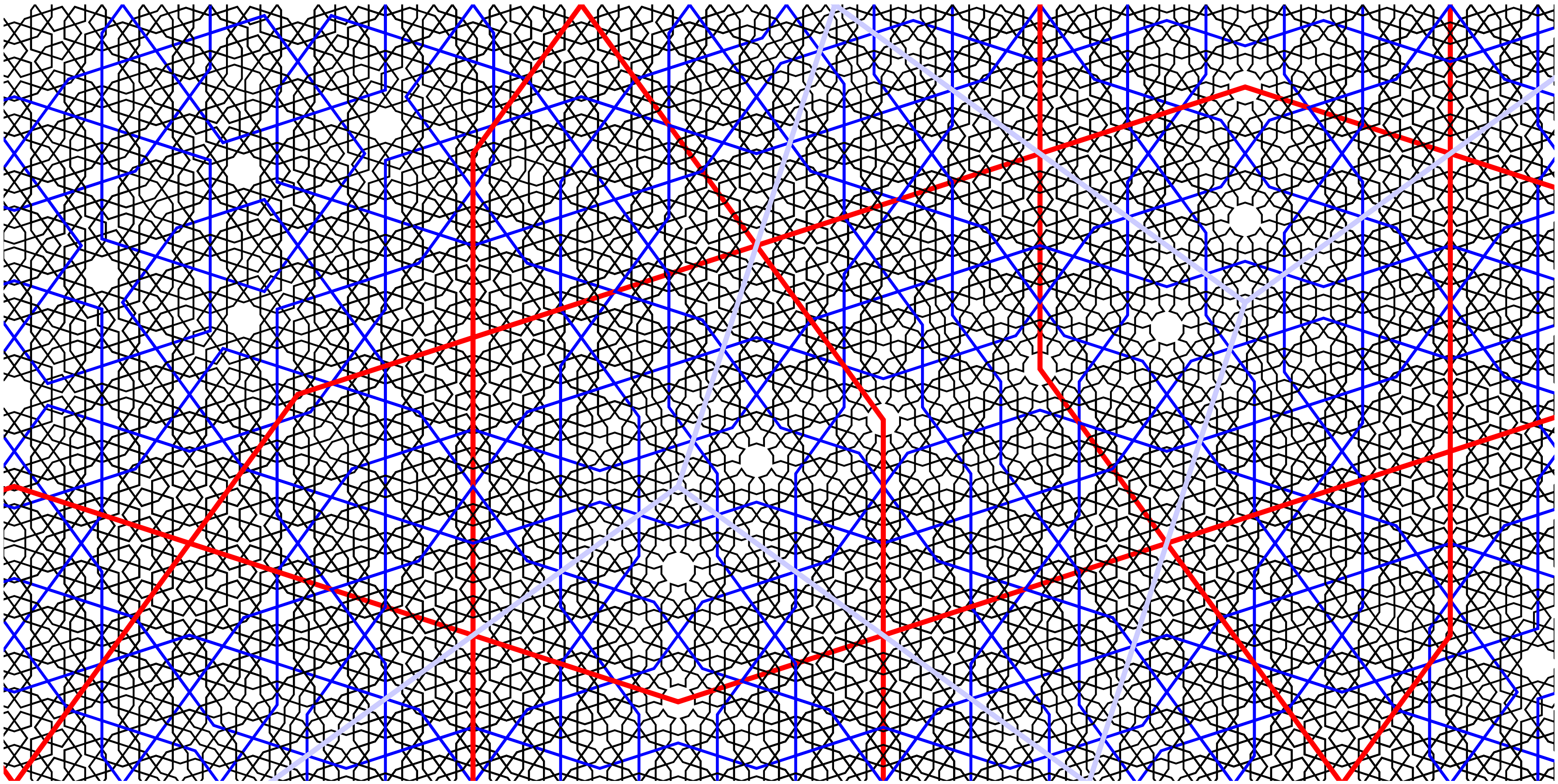


Pattern no. 28 with third level of tiles



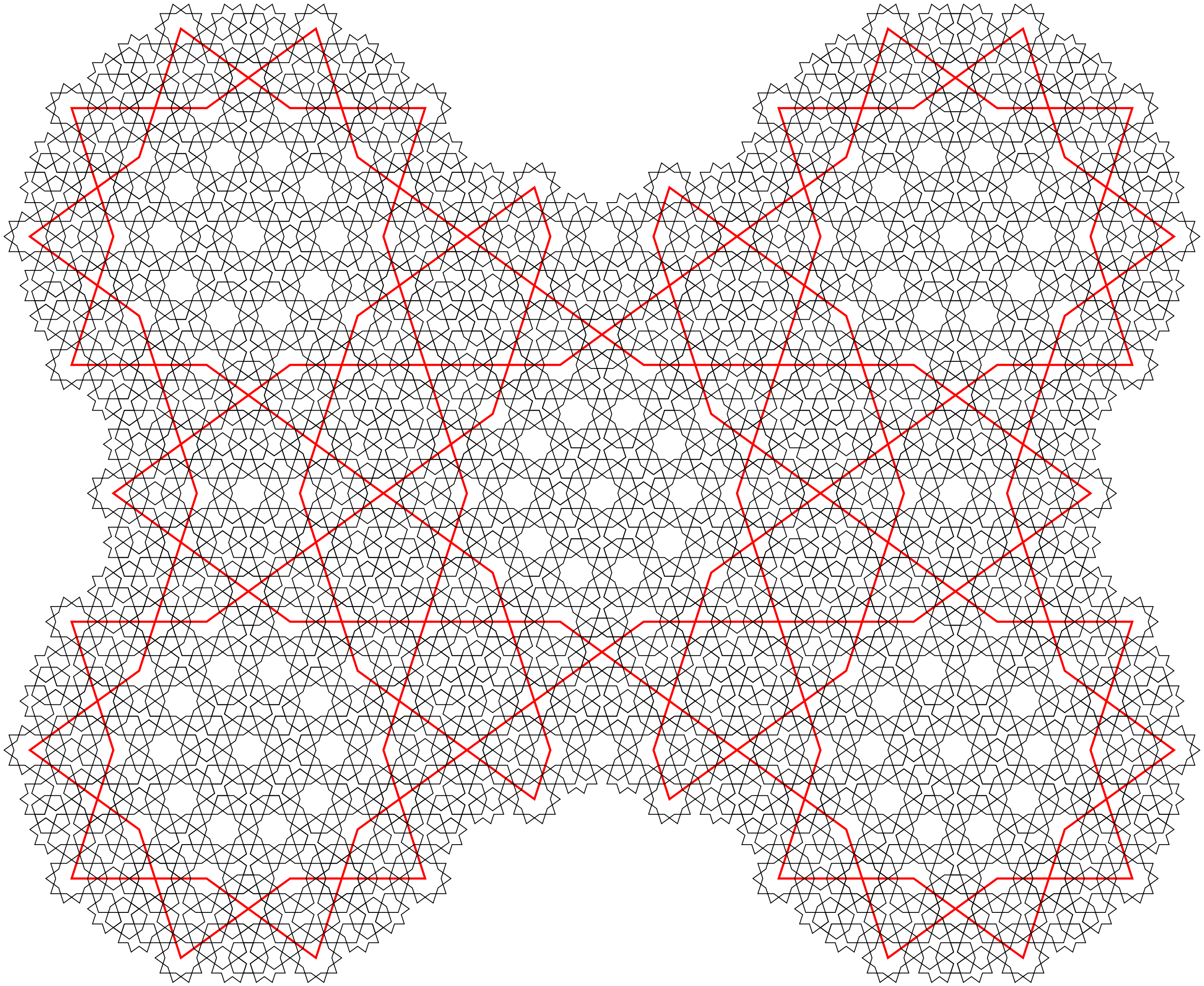
The white stars are the defects in the subdivision of the pentagons

Pattern no. 28 with third level of line ornament

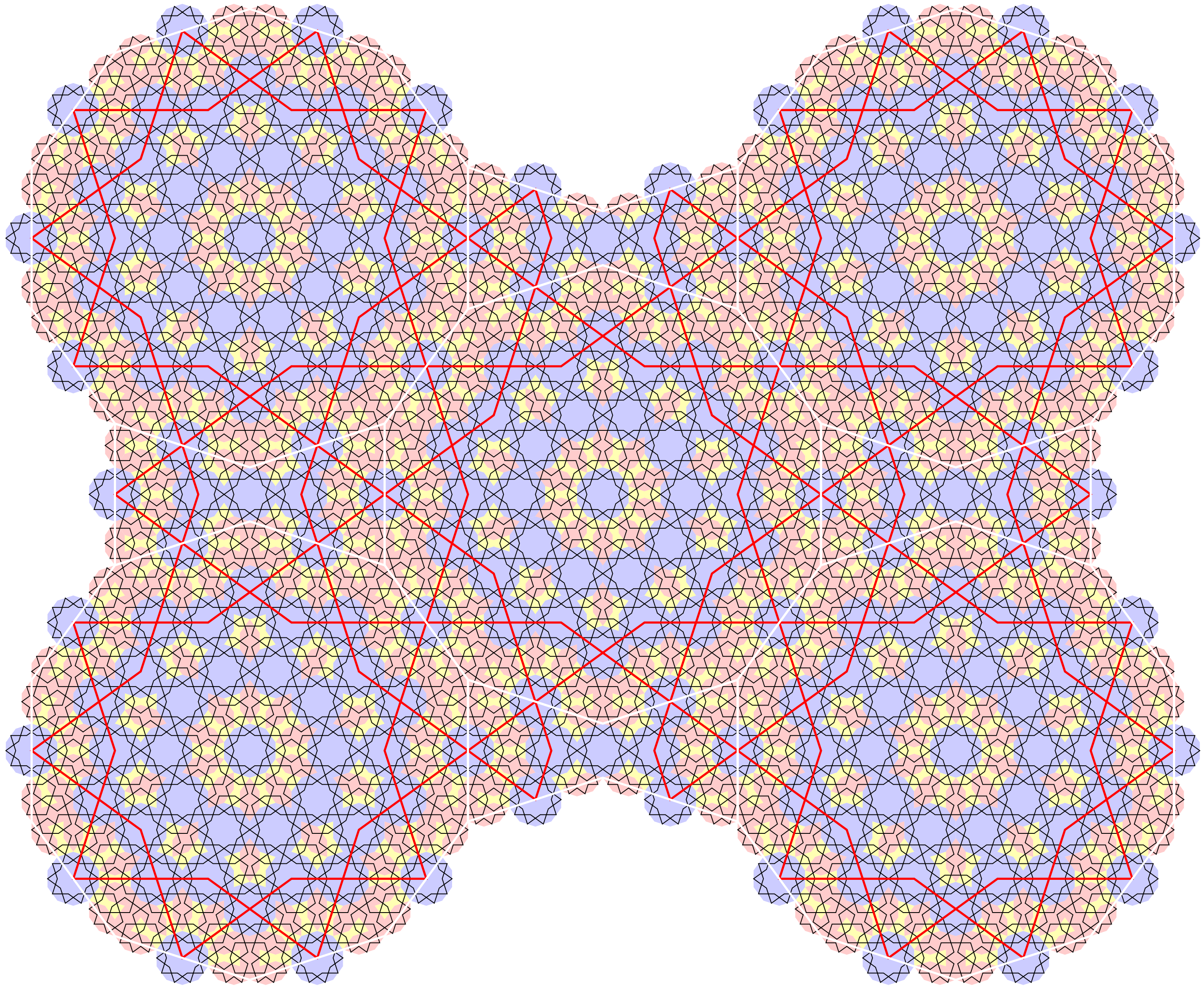


Darb-i Imam shrine in Isfahan (1453)

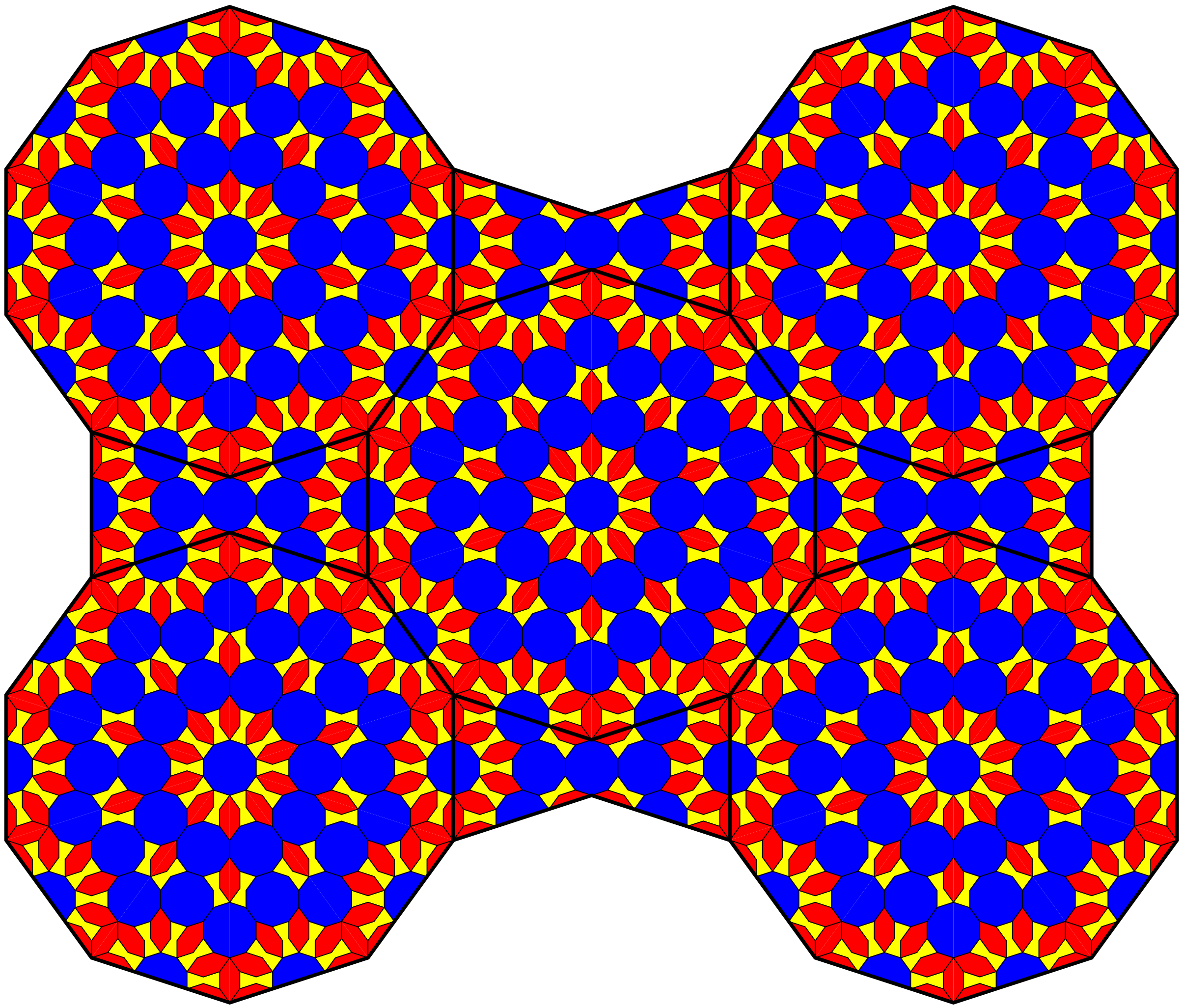
Girih pattern on two scaling levels



Girih pattern and underlying tiles on two scaling levels

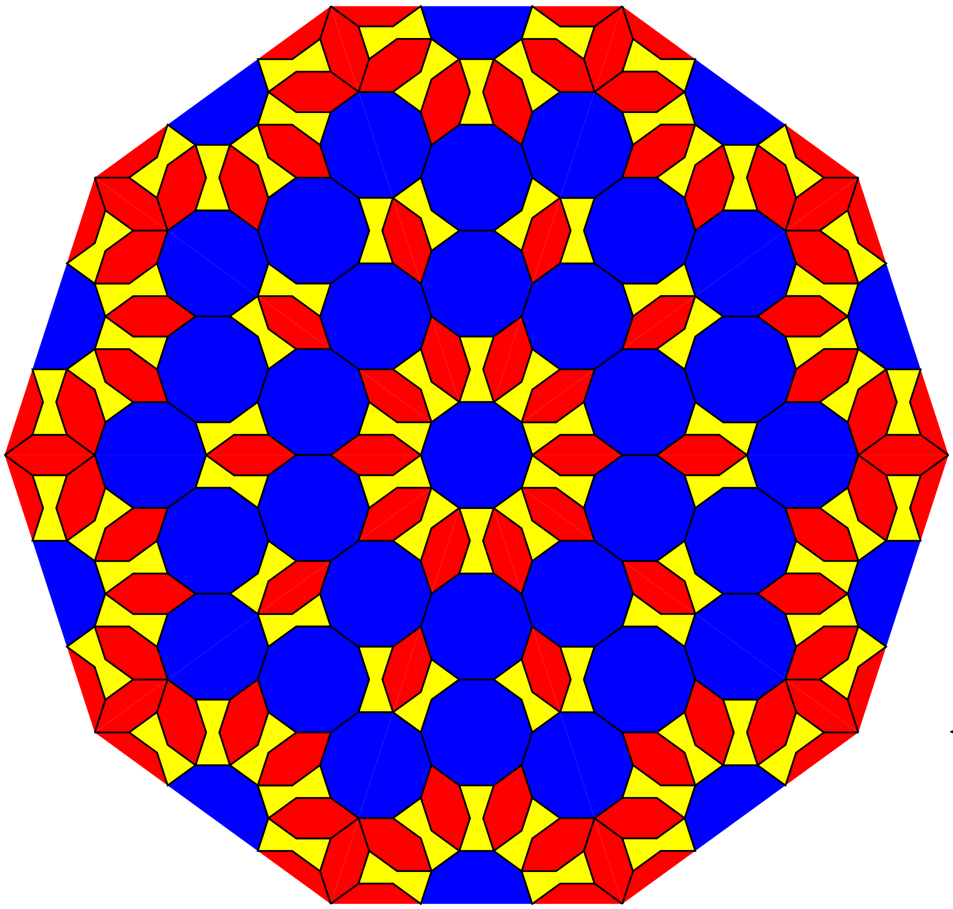


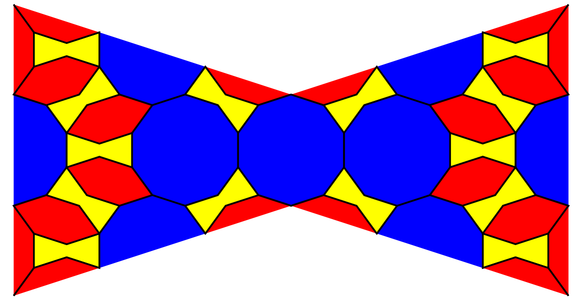
Girih tiles on two scaling levels

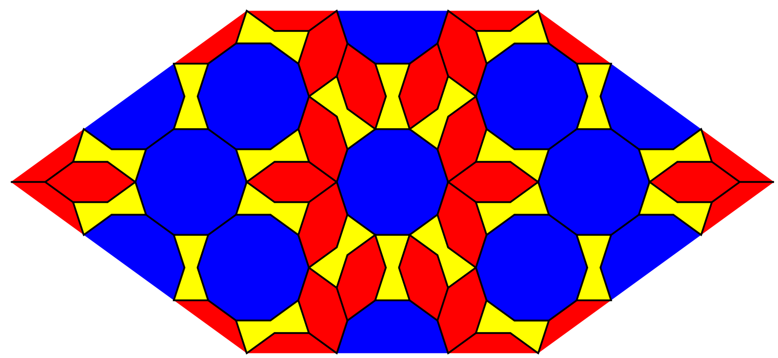


no large-scale hexagon occurs

Subdivision of girih tiles

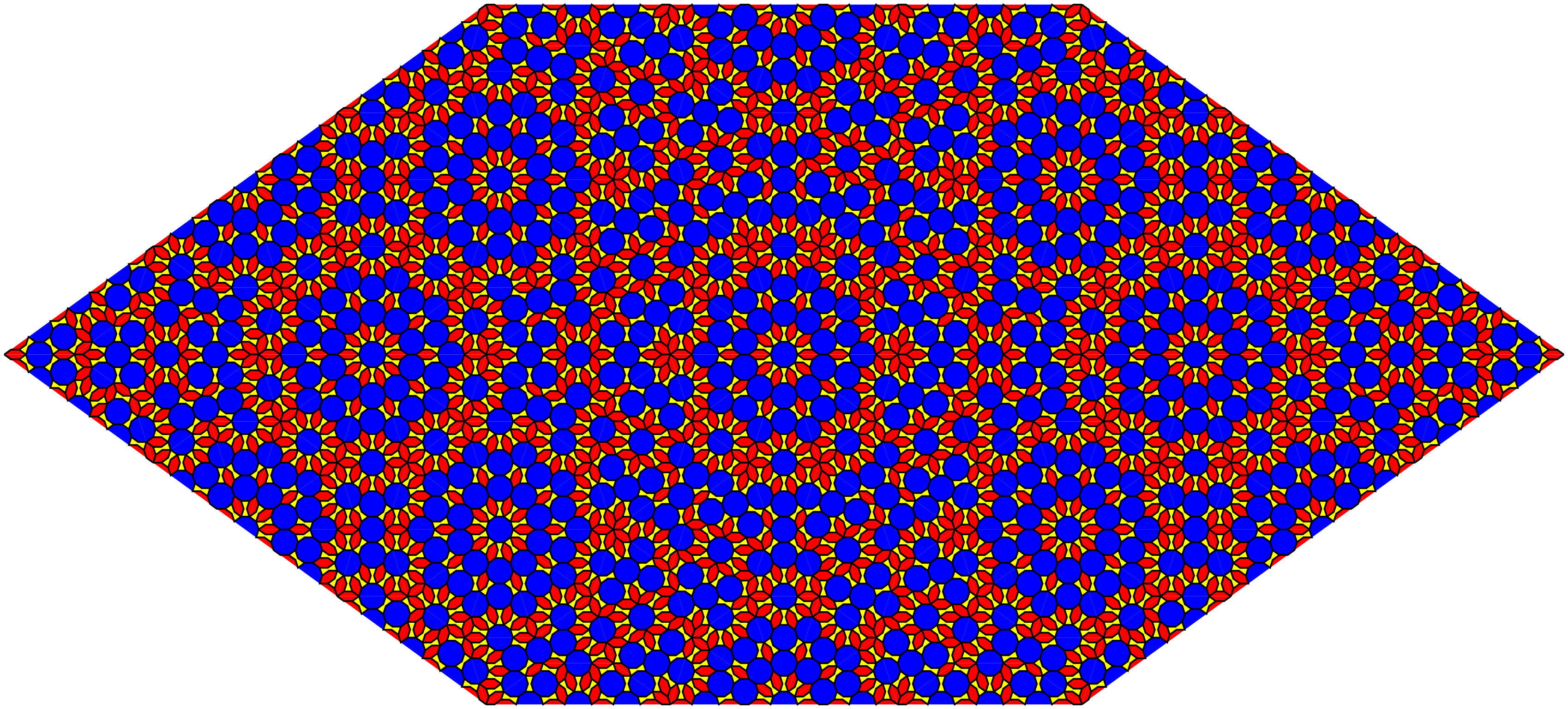




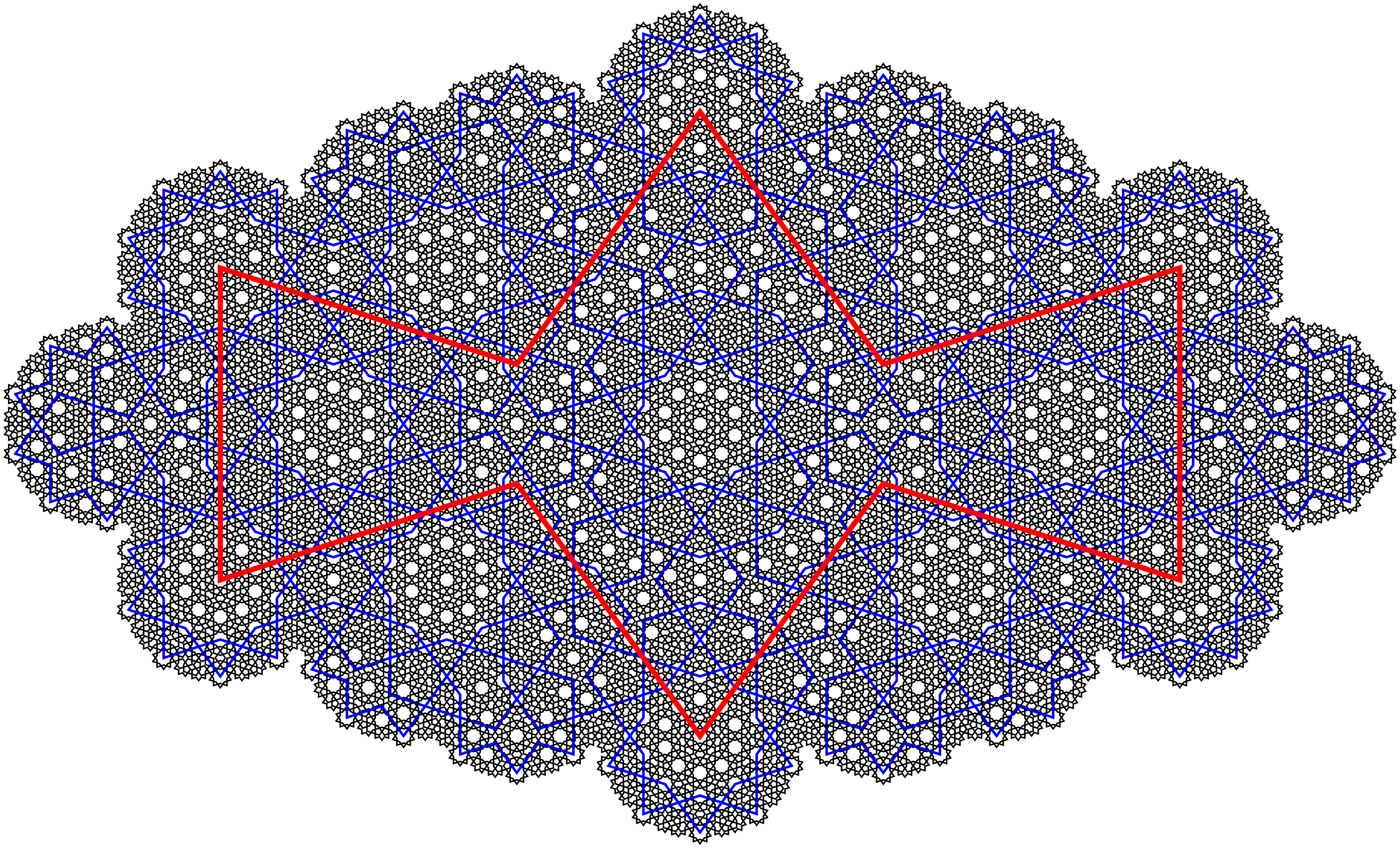


Scaling factor $4\tau + 2 \approx 8.472$

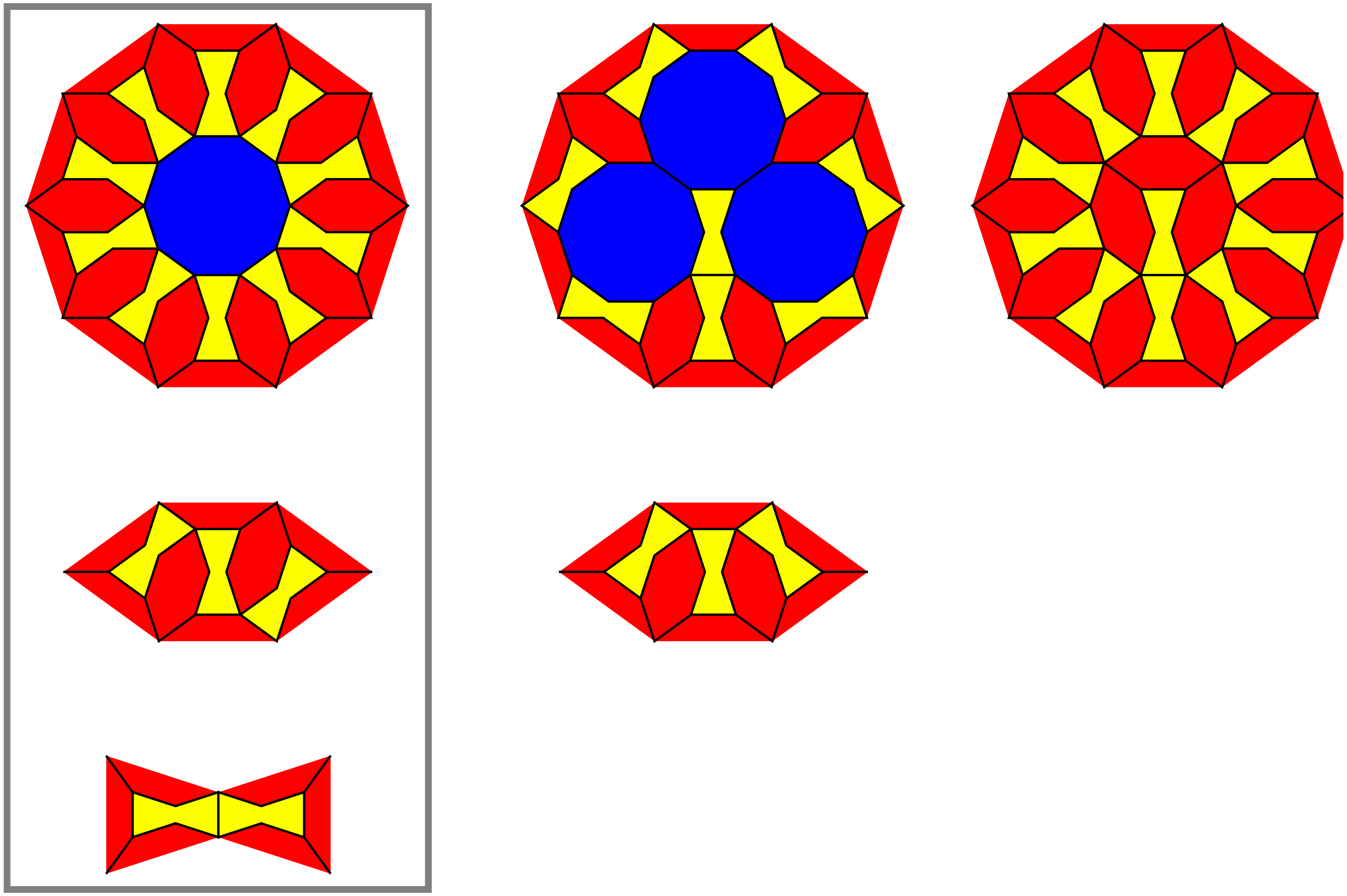
Three levels of tiles



Girih pattern on three scaling levels

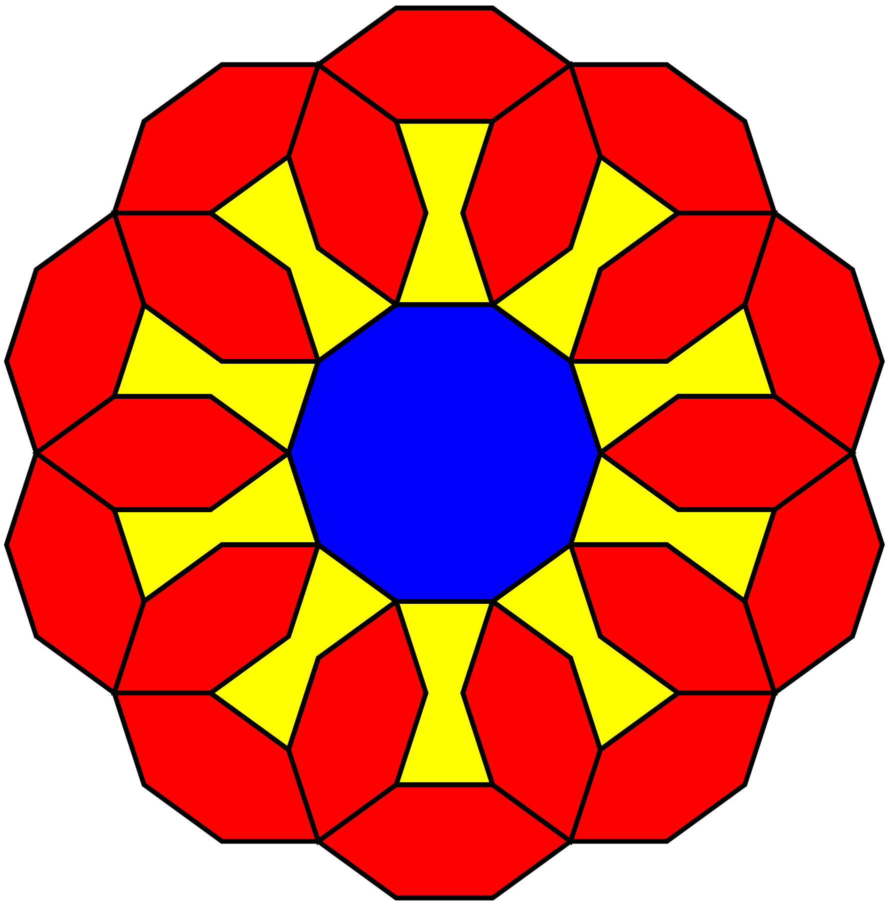


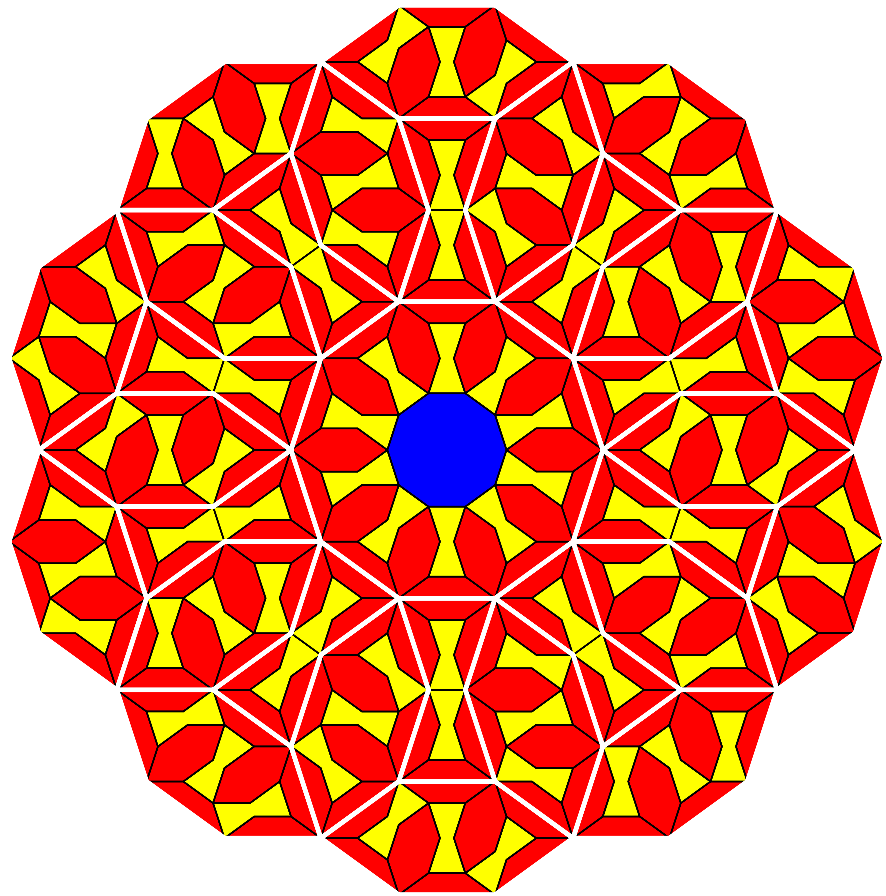
Smaller scaling factor $\longrightarrow$ more scaling levels

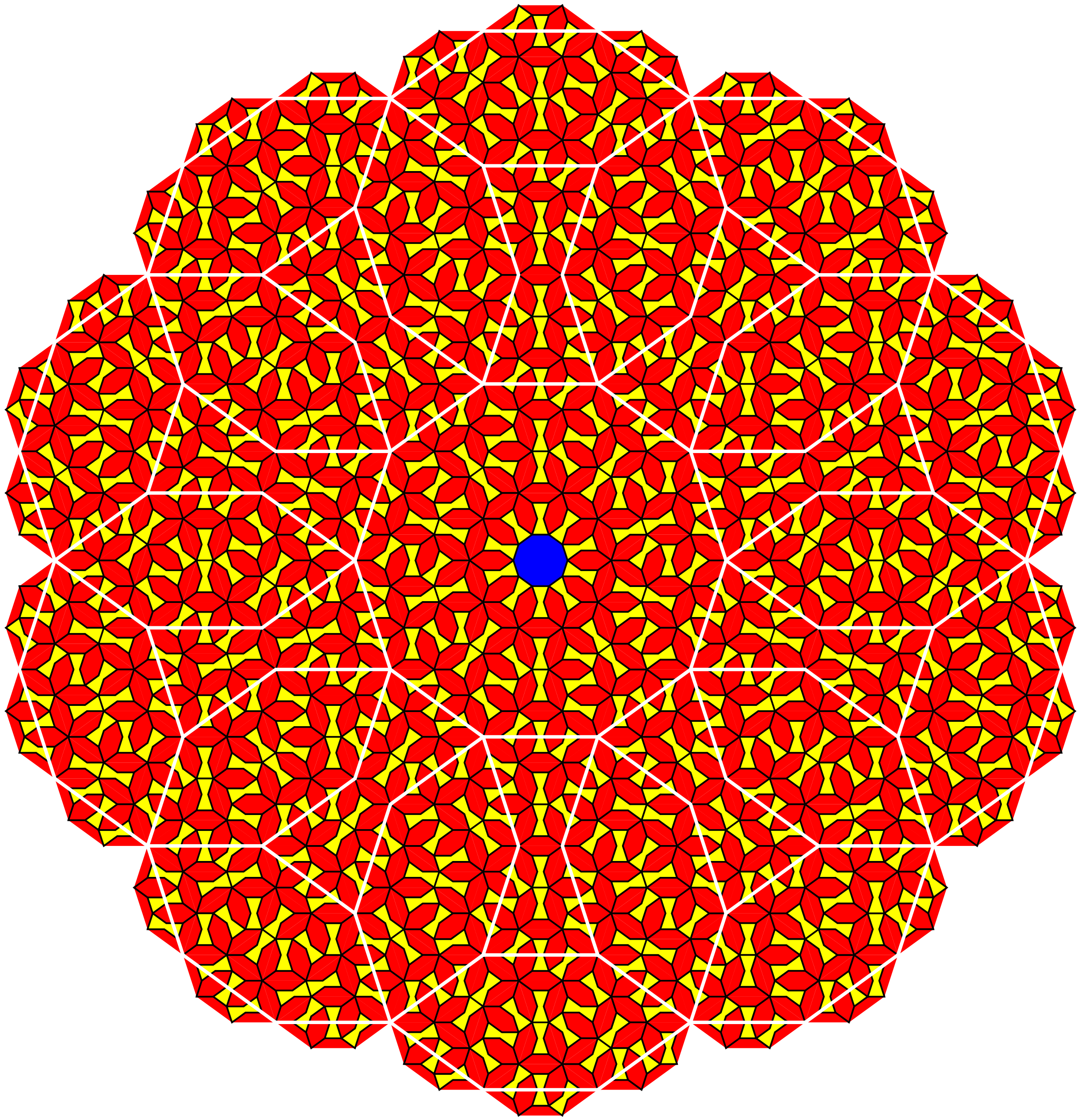


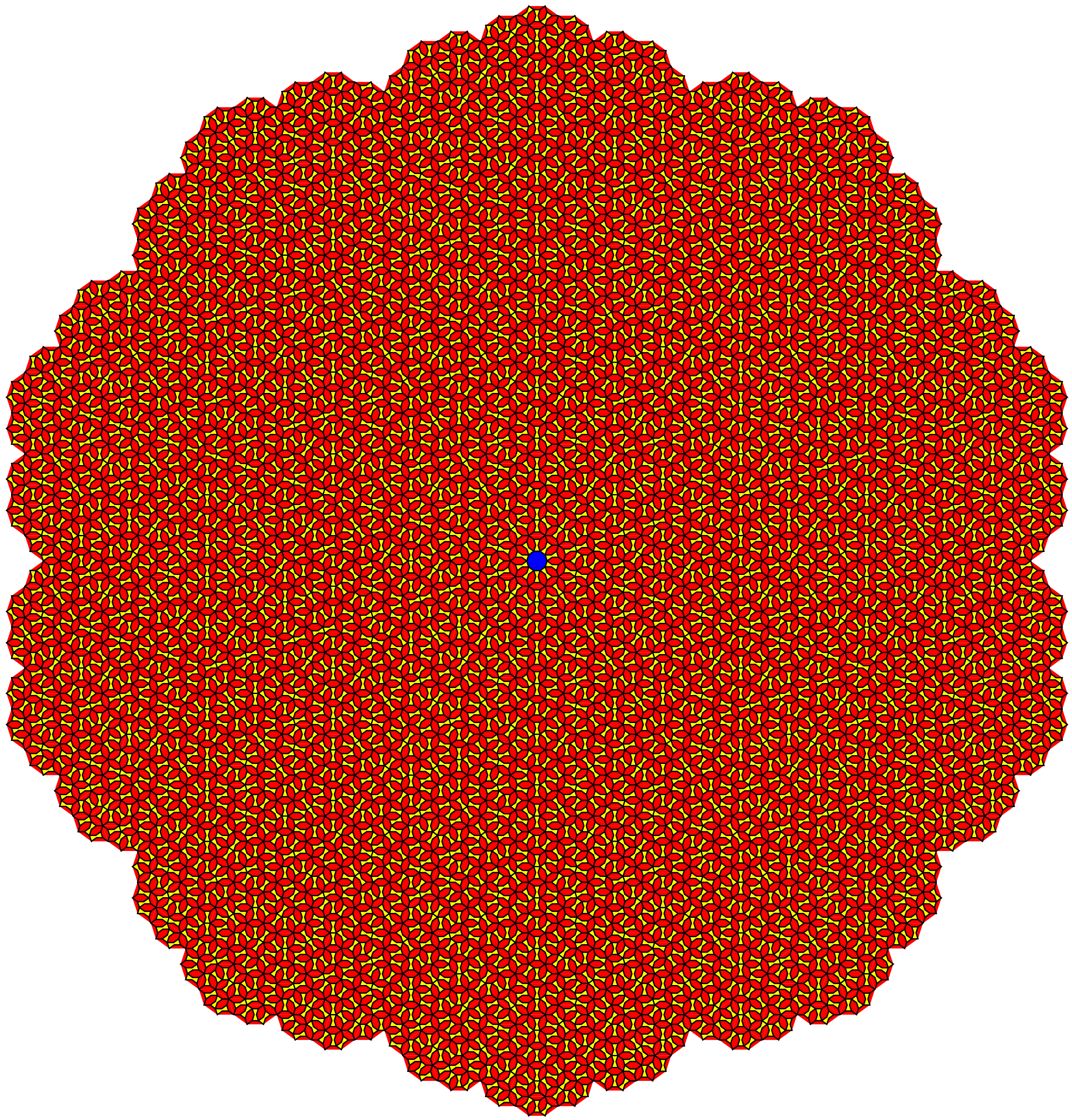
Scaling factor $\tau + 1 \approx 2.618$

Subdivision of tiles









Self-similar girih patterns on up to four scaling levels