Exercise (1). See the solution in tutorials.

Exercise (2). We have $\varphi(N)$ options for a and N options for b, because a can be any integer between 1 and N coprime to N and b can be any integer between 0 and $N-1$. Hence there are $\varphi(N)N$ encryption keys (a,b). If N is prime, then $\varphi(N)N = (N-1)N = N^2-N$. If N is composite, then the definition of $\varphi(N)$ implies that $\varphi(N) < N-1$, that is $\varphi(N)N < (N-1)N = N^2-N$, as required.

Exercise (3).
The matrix numerical equivalent of GOODLUCK is

$$\begin{pmatrix} G & O & L & C \\ O & D & U & K \end{pmatrix} \to \begin{pmatrix} 6 & 14 & 11 & 2 \\ 14 & 3 & 20 & 10 \end{pmatrix}.$$

Now

$$\begin{pmatrix} 14 & 11 \\ 17 & 10 \end{pmatrix}\begin{pmatrix} 6 & 14 & 11 & 2 \\ 14 & 3 & 20 & 10 \end{pmatrix} = \begin{pmatrix} 238 & 229 & 374 & 138 \\ 242 & 268 & 387 & 134 \end{pmatrix}$$
$$\equiv \begin{pmatrix} 4 & 21 & 10 & 8 \\ 8 & 8 & 23 & 4 \end{pmatrix} \bmod 26.$$

Replacing the entries of the last matrix by their corresponding alphabet symbols, we obtain the matrix $\begin{pmatrix} E & V & K & I \\ I & I & X & E \end{pmatrix}$.

`Answer`: EIVIKXIE.

Exercise (4).
$\begin{pmatrix} E & S \\ \sqcup & \sqcup \end{pmatrix}$ encrypts as $\begin{pmatrix} P & R \\ K & Z \end{pmatrix}$. The numerical equivalents of these matrices are $X = \begin{pmatrix} 4 & 18 \\ 26 & 26 \end{pmatrix}$ and $Y = \begin{pmatrix} 15 & 17 \\ 10 & 25 \end{pmatrix}$ respectively. Hence the inverse A^{-1} of the encryption matrix A equals $XY^{-1} \equiv \begin{pmatrix} 22 & 16 \\ 21 & 17 \end{pmatrix} \bmod 27$.

Now knowing the decryption matrix A^{-1}, we can decrypt the text. The matrix numerical equivalent of the ciphertext is

$$\begin{pmatrix} Z & I & X & V & M & P \\ R & X & Y & B & N & O \end{pmatrix} = \begin{pmatrix} 25 & 8 & 23 & 21 & 12 & 15 \\ 17 & 23 & 24 & 1 & 13 & 14 \end{pmatrix}.$$

Denote that matrix by Z. Then the matrix numerical equivalent of the plaintext is $A^{-1}Z \bmod 27$, i.e. $\begin{pmatrix} 12 & 4 & 26 & 19 & 13 & 14 \\ 4 & 19 & 0 & 26 & 14 & 13 \end{pmatrix}$. Replacing entries by corresponding letters we obtain the answer.

`Answer`: MEET AT NOON.

Exercise (5). Let us take, say, the last four letters of the ciphertext. We have that $\begin{pmatrix} A & I \\ R & A \end{pmatrix}$ encrypts to $\begin{pmatrix} D & Y \\ R & D \end{pmatrix}$, that is $\begin{pmatrix} 0 & 8 \\ 17 & 0 \end{pmatrix}$ goes to $\begin{pmatrix} 3 & 24 \\ 17 & 3 \end{pmatrix}$. Denote these matrices by X and Y respectively. Let A be the encryption matrix. Then $Y = AX$, that is the decryption matrix $A^{-1} = XY^{-1}$. Hence we have $A^{-1} \equiv \begin{pmatrix} 22 & 20 \\ 28 & 8 \end{pmatrix} \bmod 29$.

The numerical matrix equivalent of the ciphertext is

$$\begin{pmatrix} ! & W & V & E & ! & R & D & Y \\ I & G & I & X & Z & A & R & D \end{pmatrix} = \begin{pmatrix} 28 & 22 & 21 & 4 & 28 & 17 & 3 & 24 \\ 8 & 6 & 8 & 23 & 25 & 0 & 17 & 3 \end{pmatrix}.$$

Denote the last matrix by Z. Then the numerical equivalent of the plaintext is $A^{-1}Z = \begin{pmatrix} 22 & 24 & 13 & 26 & 14 & 26 & 0 & 8 \\ 7 & 26 & 14 & 6 & 27 & 12 & 17 & 0 \end{pmatrix}$. Substituting corresponding letters into the last matrix we obtain the answer.

`Answer.` WHY NO GO? MARIA.