Exercise (1)
(i) See the solution in tutorials.
(ii) See the solution in tutorials.
(iii) Use the Euclidean Algorithm to find the greatest common divisor d of $a = 6643$, $b = 2873$.

$6643 = 2873 \cdot 2 + 897$
$2873 = 897 \cdot 3 + 182$
$897 = 182 \cdot 4 + 169$
$182 = 169 \cdot 1 + 13$
$169 = 13 \cdot 13 + 0$

Thus $d = 13$.

Apply the Extended Euclidean Algorithm to find integers x, y such that $ax + by = d$, i.e. $6643x + 2873y = 13$. Reading rows of `Step 1` backwards, we obtain that

$13 = 182 - 169 \cdot 1 = 182 - (897 - 182 \cdot 4) =$
$182 \cdot 5 - 897 = (2873 - 897 \cdot 3) \cdot 5 - 897 =$
$2873 \cdot 5 + 897 \cdot (-16) =$
$2873 \cdot 5 + (6643 - 2 \cdot 2873) \cdot (-16) =$
$2873 \cdot 37 + 6643 \cdot (-16).$
`Answer.` $d = 13$; $x = -16$, $y = 37$.

Exercise (2) See the solution in tutorials.

Exercise (3) See the solution in tutorials.

Exercise (4) Find all solutions of $187x \equiv 12 \bmod 546$.
`Step 1`. Apply Euclidean Algorithm to find $gcd(187, 546)$.
$546 = 187 \cdot 2 + 172$
$187 = 172 \cdot 1 + 15$
$172 = 15 \cdot 11 + 7$
$15 = 7 \cdot 2 + 1$
$7 = 7 \cdot 1 + 0.$

Hence $gcd(187, 546) = 1$, i.e. 187 and 546 are co-prime. Hence the given equation has a unique solution.

`Step 2`. Apply the Extended Euclidean Algorithm to find a and b such that $187a + 546b = 1$. Reading rows of `Step 1` backwards, we obtain the following.

$1 = 15 - 7 \cdot 2 = 15 - (172 - 15 \cdot 11) \cdot 2 = 15 \cdot 23 - 172 \cdot 2 = (187 - 172) \cdot 23 - 172 \cdot 2 = 187 \cdot 23 - 25 \cdot 172 = 187 \cdot 23 - (546 - 2 \cdot 187) \cdot 25 = 187 \cdot 73 - 546 \cdot 25.$

Hence $187 \cdot 73 - 546 \cdot 25 = 1$, i.e. $a = 73$ is the multiplicative inverse of 187 mod 546. Thus from the original equation we have that

$x \equiv 12 \cdot 73 = 876 \equiv 330 \bmod 546.$

`Answer.` $x = 330$.

Exercise (5) See the solution in tutorials.

Exercise (6)
(i) See solutions in tutorials.
(ii) See solutions in tutorials.

(iii) $81^{119} \equiv x \bmod 13$.
By Fermat's Little Theorem $3^{12} \equiv 1 \bmod 13$. Since $81^{119} = ((3)^4)^{119} = 3^{476}$ and $476 = 39 \cdot 12 + 8$ we have that $81^{119} = 3^{476} \equiv 3^8 = 6561 = 13 \cdot 504 + 9 \equiv 9 \bmod 13$.
`Answer.` The residue equals 9.

(iv) $13^{216} \equiv x \bmod 19$. By Fermat's Little Theorem $13^{18} \equiv 1 \bmod 19$. Hence $13^{216} = 13^{18 \cdot 12} \equiv 1 \bmod 19$.
`Answer.` The residue equals 1.

Exercise (7) See the solution in tutorials.

Exercise (8) See the solution in tutorials.

Exercise (9) Consider the linear congruence $111x \equiv 1 \bmod 250$. Here x is the multiplicative inverse of 111 mod 250. To find x we apply the Extended Euclidean Algorithm (see **Exercise (4)** above). First, by the Euclidean Algorithm we have that
$250 = 111 \cdot 2 + 28$
$111 = 28 \cdot 3 + 27$
$28 = 27 \cdot 1 + 1$
$27 = 27 \cdot 1 + 0$.
Thus 250 and 111 are co-prime, and the congruence has a unique solution. Reading steps of Euclidean Algorithm backwards we have that $1 = 28 - 27 = 28 - (111 - 28 \cdot 3) = 4 \cdot 28 - 111 = 4 \cdot (250 - 2 \cdot 111) - 111 = 4 \cdot 250 + 111 \cdot (-9)$.
Hence $x = -9 \equiv 241 \bmod 250$.
`Answer.`The multiplicative inverse 111 mod 250 equals 241.

Exercise (10) See the solution in tutorials.

Exercise (11) Fermat's Little Theorem implies that $2^{10} \equiv 1 \bmod 11$. Since $11213 = 10 \cdot 1121 + 3$ we have that $2^{11213} \equiv 2^3 \bmod 11$. Hence $2^{11213} - 1 \equiv 7 \bmod 11$, i.e. $2^{11213} - 1$ is not divisible by 11.