Groups I MA343/MA532 Problems I

1. Calculate the *k*-th power a^k of the matrix $a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ for $k = 2,3,4$.

   Show that $G = \{I, a, a^2, a^3\}$ is a group w.r.t. the usual matrix multiplication. Does (the table for) this group remind you of one that you have seen before?

2. Calculate the table of the symmetric group Sym(3) of all permutations of $\{1,2,3\}$. List the elements in the order id, $\alpha, \beta, \rho_1, \rho_2, \rho_3$ where $\alpha = (123)$, $\beta = \alpha^2$, $\rho_1 = (23)$ etc.

3. Consider the matrices $a = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ and $c = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Calculate $b = a^2$, $d = ac$ and $e = bc$. Show that $G = \{I, a, b, c, d, e\}$ is a group under matrix multiplication. For each *x* in *G*, determine the (elements of) the cyclic subgroup <*x*> generated by *x*. Is *G* itself a cyclic group? Why (not)?

4. The binary operation $\bullet$ is defined on $G = \{1,2,3,4\}$ by $a \bullet b = \max\{a,b\}$. For example, $2 \bullet 4 = \max\{2,4\} = 4$. Determine which group axioms are satisfied by *G* w.r.t. $\bullet$.

5. Show that each of $G = \{I, -I, a, -a\}$ and $H = \{I, -I, b, -b\}$ is a group under matrix multiplication, where $a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Show that one of these groups is cyclic and that the other is not.

6. Let $G = \{1, a, b, c\}$ be a multiplicative group, with 1 as identity element. Show that
   either **exactly one** of a, b or c is its own inverse (say a)
   or **each** of a, b or c is its own inverse.
   Deduce that there are just two ways to complete the table shown, and so there are just two (essentially) different groups of order 4.

	1	*a*	*b*	*c*
1	1	*a*	*b*	*c*
a	*a*	1	.	.
b	*b*	.	.	.
c	*c*	.	.	.

7. Show that *G* = (-1,1) is a group under $\bullet$ defined by $a \bullet b = \dfrac{a+b}{1+ab}$.
   [For closure, show that if $a, b \in (-1,1)$ then $(a+b)^2 < (1+ab)^2$.]

****Homework: 2, 5, 6. Due: 28 September. ****