

ON THE BASE SIZE OF A TRANSITIVE GROUP WITH SOLVABLE POINT STABILIZER

Evgeny Vdovin¹

¹Sobolev Institute of mathematics SB RAS

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Thus there exists a homomorphism $N_G(A/B) \rightarrow \text{Aut}(A/B)$. The image of $N_G(A/B)$ under this homomorphism is denoted by $\text{Aut}_G(A/B)$ and is called a *group of induced automorphisms* of A/B .

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We always assume that G acts on $G : H$ by right multiplication. Clearly, the action of G on x^G “is the same” as the action of G on $G : G_x$ by right multiplication $(G_x\tau)\sigma = G_x(\tau\sigma)$. Clearly, H_G is the kernel of the action of G on $G : H$.

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If G acts on X , then the action of G on X^k is defined by

$$\sigma : (x_1, \dots, x_k) \mapsto (x_1^\sigma, \dots, x_k^\sigma).$$

We consider the following problem

Main problem

Given G acting transitively on X find the minimal k such that there exist points, $x_1, \dots, x_k \in X$, with $\forall i = 1, \dots, k \quad x_i \sigma = x_i \Rightarrow \forall x \in X, x \sigma = x$. This set of points x_1, \dots, x_k is called a base of G , while the number k is called the base size.

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Using the fact that for every $x \in X$ and $y = x\sigma$ the identity $G_y = G_x^\sigma := \sigma^{-1} G_x \sigma$ holds, the main problem can be formulated in the following way:

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Given subgroup H of G find the minimal number k of conjugated with H subgroups such that their intersection equals H_G .

Using the action of G on X^k we may give another equivalent form of the main problem.

Main problem

Given subgroup $H \leq G$ consider the action of G on $X = G : H$ by right multiplication. Find the minimal k such that X^k possesses a regular G/H_G -orbit.

The number k from the main problem is called the *base size* of G with respect to H , and is denoted by $\text{Base}_H(G)$. If $G \leq \text{Sym}(X)$ is transitive, then put $\text{Base}(G) = \text{Base}_{G_x}(G)$, where $x \in X$. The number of regular G/H_G -orbits on X^k is denoted by $\text{Reg}_H(G)$.

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Let again $X = G : H$ and $k = \text{Base}_H(G)$. If (x_1, \dots, x_k) is a G -regular point, then, clearly,

$(x_1, \dots, x_k)^G \leq |X| \cdot (|X| - 1) \cdot \dots \cdot (|X| - k + 1) < |X|^k$. Since $|X| = |G : H|$, we obtain $|G/H_G| < |G : H|^k$ and, equivalently $|H/H_G| < |G : H|^{k-1}$.

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Index bounding problem

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- 1 $L = \text{Alt}_n$ and X is an orbit under the action on subsets or partitions of $\{1, \dots, n\}$.
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Theorem (M.W.Liebeck, A.Shalev, 2005).

Let G be a finite almost simple group having socle a classical group with natural module of dimension greater than 15. Suppose G acts primitively on a set X in a non-subspace action. Then the probability that three randomly chosen points in X form a base for G tends to 1 as $|G| \rightarrow \infty$. In particular, for G sufficiently large we have $\text{Base}(G) \leq 3$.

Theorem (T.C.Burness, 2007).

Let G be a finite almost simple classical group in a faithful primitive nonstandard action. Then either $\text{Base}(G) \leq 4$, or $G = U_6(2) \cdot 2$, $H = U_4(3) \cdot 2^2$ and $\text{Base}_H(G) = 5$.

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Theorem (T.C.Burness, M.W.Liebeck, A.Shalev, 2009).

Let G be a finite almost simple group in a primitive faithful non-standard action. Then $\text{Base}(G) \leq 7$, with equality if and only if G is the Mathieu group M_{24} in its natural action of degree 24.

Theorem (E.Vdovin)

Let G be a group and let

$$\{e\} = G_0 < G_1 < G_2 < \dots < G_n = G$$

be a composition series of G that is a refinement of a chief series. Assume that for some k the following condition holds: If G_i/G_{i-1} is nonabelian, then for every solvable subgroup T of $\text{Aut}_G(G_i/G_{i-1})$ we have

$$\text{Base}_T(\text{Aut}_G(G_i/G_{i-1})) \leq k \text{ and } \text{Reg}_T(\text{Aut}_G(G_i/G_{i-1}), k) \geq 5.$$

Then, for every maximal solvable subgroup S of G , we have $\text{Base}_S(G) \leq k$.

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Then, for every maximal solvable subgroup S of G , we have $\text{Base}_S(G) \leq k$.

Is it possible to remove condition $\text{Reg}_S(\text{Aut}_G(G_i, G_{i-1}), k) \geq 5$?

Example 1 (V.I.Zenkov, 2008)

Consider $G = \text{Sym}_5 \wr \text{Sym}_2$ and $S = \text{Sym}_4 \wr \text{Sym}_2$. It is evident that Alt_5 is the unique nonabelian composition factor of G (however there are two nonabelian composition factors isomorphic to Alt_5). It is also easy to see, that for every solvable subgroup T of $\text{Sym}_5 = \text{Aut}(\text{Alt}_5)$ we have $\text{Base}_T(\text{Sym}_5) \leq 4$. However $\text{Reg}_{\text{Sym}_4}(\text{Sym}_5, 4) = 1$ and $\text{Base}_S(G) = 5$.

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Example 2 (E.Vdovin, V.I.Zenkov, 2009)

Consider $G = \text{Sym}_8$ and $S = \text{Sym}_4 \wr \text{Sym}_2$. Then $\text{Base}_5(G) = 5$. Notice also that the inequality $\text{Reg}_5(G, 5) \geq 12$ is proven.

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Lemma (E.Vdovin)

Let G be a transitive permutation group acting on $X = \{1, \dots, n\}$ and let the stabilizer S of 1 be solvable. Assume that $k = \max\{\text{Base}(G), 6\}$. Then $\text{Reg}(G, k) \geq 5$.

Theorem (L.Babai, A.J.Goodman, L.Pyber, 1997).

There exists a constant c such that every G possessing a solvable subgroup of index n possesses a normal solvable, subgroup of index at most n^c .

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Kourovka notebook, 17.41

Let S be a maximal solvable subgroup of a finite group G .

- (a) Is it true that $\text{Base}_S(G) \leq 7$?
- (b) Is it true that $\text{Base}_S(G) \leq 5$?

Kourovka notebook, 15.40

Let N be a nilpotent subgroup of a finite simple group G . Does there exist $\sigma \in G$ such that $N \cap N^\sigma = \{\varepsilon\}$?

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Notice that Problem 17.40 is not equivalent to the main problem in case of nilpotent point stabilizer. Moreover, the known examples show that if $N \leq G$ and N is nilpotent, then $\text{Base}_N(G)$ can be unbounded even if G is solvable.

“Problem”

Let G be a group and let

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be a composition series of G that is a refinement of a chief series. Consider $H \leq G$ and for every nonabelian G_i/G_{i-1} denote $(H \cap N_G(G_i/G_{i-1}))G_{i-1}/G_{i-1}$ by H_i . Assume that there exists k such that for every nonabelian G_i/G_{i-1} we have $\text{Base}_{H_i}(\text{Aut}_G(G_i/G_{i-1})) \leq k$. Can we bound $\text{Base}_H(G)$ by a function $f(k)$?