On the generalized octagon of order $(2, 4)$

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5th De Brún Workshop, 12th April 2011

Generalized polygons

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A *generalized n-gon* of order (s, t) is a point-line geometry $(P, \mathcal{L})$ satisfying the following properties:

1. diameter of the incidence graph Γ is n;
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A generalized polygon is *thick* if $s,t \geq 2$. By the Feit-Higman Theorem, a finite thick generalized n-gon can only exist for $n = 2,3,4,6$ and 8.

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- The smallest generalized triangle has order $(2, 2)$ and it is the Fano plane, the unique projective plane of order two. Γ has 7 points and 7 lines. Its automorphism group is $L_3(2)$.

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- The smallest generalized hexagon is of order $(2, 2)$, yielding 63 points and 63 lines. There is an example coming from the group $G_2(2)$. The uniqueness (up to duality) is due to Tits (1959) and Cohen and Tits (1985).

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The uniqueness of the generalized octagon of order $(2, 4)$ is an open question.

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Can we determine all such octagons $\mathcal{O}$?

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Since 8 is even, far away objects are of the same type, two points or two lines.

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Corollary

Any two points, x and y, that are far away from each other are connected by exactly $5 = t + 1$ shortest paths in Γ, one through each line on x and, symmetrically, through each line on y.

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Any two points, x and y, that are far away from each other are connected by exactly $5 = t + 1$ shortest paths in Γ, one through each line on x and, symmetrically, through each line on y.

This establishes a matching between the lines on x and on y.

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For any two lines, L and R, for away from each other there is a matching between the points on L and on R, defined by the shortest paths between L and R.

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Our first step is the following.

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Q acts on X regularly, in particular, $|Q| = 1024 = 2^{10}$.

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In this way, every edge in Δ get one of the five colours, and every vertex of Δ lies on exactly one edge of each colour.

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Unfortunately, we do not know that Δ is connected. Even worse, it is disconnected in the known example.

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□

Thus, Q is a 2-group of order 2^{10} and it is regular on X.

Five involutions

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For information, there are $56,092$ groups of order 2^8 and $10,494,213$ groups of order 2^9. The complete list is available in and MAGMA and GAP, due to O'Brien.

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Corollary

Q_0 induces on the ten points an elementary abelian group of order 2^4. Namely, the element $\alpha_1 \cdot \ldots \cdot \alpha_5$ is in the kernel of the action.

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Based on this observation and using the condition that the girth of the Cayley graph should be at least eight, it was possible to compile a short complete list of groups that can be generated by three of the five involutions α_i.

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A few hundred groups survived the check, so these required additional criteria to dispose of them.

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The additional condition came from the following observation.

Lemma

For each $1 \leq i \leq 5$*, there exists a set of four edges* $\{e_1 = xy_i, e_2, e_3, e_4\}$ *of* Δ*, all of the same colour* i*, such that the pairwise distance between them is at least* 6*.*

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This is an extremely strong condition going half way towards recovering $\mathcal{O}$ from Δ, and in fact it kills all the wrong groups. That is, only the group coming from the example, the Tits' octagon, survives this check.

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Consider the cycle of length 16 (an apartment) in Γ through a and x, that passes through lines (colours) 1 and 2. Let this be the cycle

$$ab_1b_2b_3b_4b_5b_6b_7xc_7c_6c_5c_4c_3c_2c_1a,$$

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Let A_j, $1 \leq j \leq 7$, be the joint stabilizer in Q of c_j and b_{8-j}.

Lemma

The order of A_j is two if c_j and b_{8-j} are lines, and it is four if c_j and b_{8-j} are points.

Proof.

There are two (respectively, four) apartments completing the half apartment from c_j to b_{8-j} via a, if c_j is a line (respectively, point). □

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Lemma

We have $Q = A_1 A_2 A_3 A_4 A_5 A_6 A_7$.

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Thus, A_1, A_3, A_5, and A_7 have order two, while A_2, A_4, and A_6 have order four.

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The subproduct from A_i to A_j, $i \leq j$, is simply the joint stabilizer Q_{ij} in Q of c_i and b_{8-j}. (In particular, $Q = Q_{17}$ and $Q_{jj} = A_j$.)

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The proof proceeds by "induction" on length, using the equalities $Q_{i-1,j} = A_{i-1}Q_{ij}$ and $Q_{i,j+1} = Q_{ij}A_{j+1}$. □

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Unfortunately, there is a great many different factorizations (of the same groups). By crude estimates, the bound is 10^{40} or so. Based on later calculations, the real number must be somewhere between tens of billions and low trillions.

The real issue is that it is difficult to formulate a strong cutoff condition that does not involve the entire factorization, from A_1 to A_7.

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The list consists of about 3 million factorizations; there are some repeatitions, but not too many. The only real concern was the completeness of the list.

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The automorphism α_1 normalizes the series $A_2 < A_2A_3 < A_2A_3A_4 < A_2A_3A_4A_5$, so it lies in the normalizer N_1 of this series in $Aut\, R$. Similarly, α_2 lies in the normalizer N_2 of the series $A_3A_4A_5A_6 > A_4A_5A_6 > A_5A_6 > A_6$.

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So the choices are somewhat limited, although N_1 and N_2 can still be very big. However, the best would be not to choose α_1 and α_2 at all.

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This simple observation kills close to 90% of all factorizations.

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We can select the pair α_1, α_2 up to conjugation by the automorphism group of the factorization (the normalizer in $Aut\, R$ of the five factors A_i). This group tends to be bigger in the bad cases.

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This shaves another order of magnitude off the length of our list.

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Once all selections are made, we have $Q = Q_0$, together with the five involutions α_i. The condition that the girth of Δ is at least eight amounts to checking that none of about a hundred particular words in $\alpha_1, \ldots, \alpha_5$ of length seven or less results in the identity element.

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Theorem

Up to isomorphism, there is only one generalized octagon of order $(2, 4)$*, the Tits' octagon, satisfying the Hypothesis.*