

Balanced tripartite entanglement, the alternating group A_4 and the Lie algebra $sl(3, \mathbb{C}) \oplus u(1)$

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*The 3rd de Brún Workshop: Algebra, Algorithms,
Applications: 30 Nov. to 10 Dec. 2009*

- ▶ Entangled **three-qubit** states of the GHZ, W , and **balanced** families (B -type)
- ▶ Realization of the largest Weyl group $W(E_8)$ from three-qubit gates
- ▶ **Condensation** of $W(E_8)$ to A_4 from B -state encoding: Lie algebra $sl(3, \mathbb{C}) \oplus u(1)$

Applications to particle physics and to the genetic code.

¹0912.0172 (math-ph)

Exchange matrices 1

► **Qubits** $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1.$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_x|0\rangle = |1\rangle, \quad \sigma_x|1\rangle = |0\rangle.$$

► **Two-qubits and the Swap gate**

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |0\rangle \otimes |0\rangle, \quad |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |0\rangle \otimes |1\rangle \dots$$

$$\text{Swap} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{Swap}|01\rangle = |10\rangle, \quad \text{Swap}|10\rangle = |01\rangle.$$

► **Two-qubits and the $Cnot$ gate**


$$Cnot = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad Cnot |10\rangle = |11\rangle, \quad Cnot |11\rangle = |10\rangle.$$

Generation of **entangled states**:

$$Cnot(\alpha |0\rangle + \beta |1\rangle) |0\rangle = a |00\rangle + b |11\rangle,$$

► **Three-qubits and the Toffoli gate.**

$$Tof = C^2 not = \begin{pmatrix} 1 & 0 \\ 0 & Cnot \end{pmatrix}, \quad Tof |110\rangle = |111\rangle, \quad Tof |111\rangle = |110\rangle.$$

²Nielsen & Chuang *Quant. Comp. and Quant. Inf.* Cambridge Press (2000) 

- ▶ Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- ▶ Mermin's square

$$\sigma_x \otimes \sigma_x \quad \sigma_y \otimes \sigma_y \quad \sigma_z \otimes \sigma_z$$

$$\sigma_y \otimes \sigma_z \quad \sigma_x \otimes \sigma_z \quad \sigma_x \otimes \sigma_y$$

$$\sigma_z \otimes \sigma_y \quad \sigma_z \otimes \sigma_x \quad \sigma_y \otimes \sigma_x$$

- ▶ Four-dim Kochen Specker theorem

$$\prod \text{eigenvalues} = 1 \quad \text{and} \quad \prod \text{observables} = -I$$

³Mermin N D 1993 *Rev. Mod. Phys.* **65** 803,

Planat M and Saniga M 2008 *Quant. Inf. Comp.* **8** 127.

Two Mermin real triples⁴

Two Mermin's triples of (mutually commuting and **real**)

$\{\sigma_x \otimes \sigma_x, \sigma_y \otimes \sigma_y, \sigma_z \otimes \sigma_z\}$ and $\{\sigma_x \otimes \sigma_z, \sigma_z \otimes \sigma_x, \sigma_y \otimes \sigma_y\}$.

- ▶ Joined eigenstates of the first triple as the rows of a orthogonal matrix (**braiding matrix**)

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} + & + & - \\ - & - & - \\ - & + & + \\ + & - & + \end{pmatrix}.$$

- ▶ Joined eigenstates of the second triple as the rows of the entangling orthogonal matrix (**B -matrix**)

$$S = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} + & - & - \\ - & + & - \\ - & - & + \\ + & + & + \end{pmatrix}.$$

⁴Planat M 2009 *Clifford group dipoles and the enactment of Weyl/Coxeter group E_8 by entangling gates*. Preprint 0904.3691 (quant-ph).

$$RS = H \otimes I \text{ with } H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

$$G_{96} = \langle R, S \rangle \cong U_{13} \cong \mathbb{Z}_4 \cdot S_4,$$

Smallest degree invariant of U_{13} :

$$\mathcal{W} := \alpha^8 + 14\alpha^4\beta^4 + \beta^8.$$

Smallest degree invariant of G_{96} :

$$W^{(2)} := \sum_{i=1}^4 \alpha_i^8 + 14 \sum_{j>i} \alpha_i^4 \alpha_j^4 + 168 \prod_{i=1}^4 \alpha_i^2,$$

⁵Nebe G, Rains E M and Sloane N J A 2006

► Measuring **two-qubit entanglement**

Tangle $\tau = C^2$ with **concurrence** $C(\psi) = |\langle \psi | \tilde{\psi} \rangle|$.

Flipped transformation $\tilde{\psi} = \sigma_y |\psi^*\rangle$

Filpped density matrix $\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)$

Taking the eigenvalues λ_i of $\rho \tilde{\rho}$ in decreasing order

$$C(\rho) = \max \left\{ 0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4} \right\},$$

► Let $|\psi\rangle = \alpha |00\rangle + \beta |01\rangle + \gamma |10\rangle + \delta |11\rangle$,

$$C = 2 |\alpha\delta - \beta\gamma|, \quad 0 \leq C \leq 1,$$

Thus $C = 0$ *separable* and $C = 1$ *maximally entangled*.

⁶W. K. Wootters, PRL 80, 2245 (1998)

Three-tangle of a three-qubit state ⁷

$$|\psi\rangle = \sum_{a,b,c=0,1} \psi_{abc} |abc\rangle,$$

SLOCC invariant three-tangle: $\tau^{(3)} = 4 |d_1 - 2d_2 + 4d_3|,$

$$d_1 = \psi_{000}^2 \psi_{111}^2 + \psi_{001}^2 \psi_{110}^2 + \psi_{010}^2 \psi_{101}^2 + \psi_{100}^2 \psi_{011}^2,$$

$$d_2 = \psi_{000} \psi_{111} (\psi_{011} \psi_{100} + \psi_{101} \psi_{010} + \psi_{110} \psi_{001}) \\ + \psi_{011} \psi_{100} (\psi_{101} \psi_{010} + \psi_{110} \psi_{001}) + \psi_{101} \psi_{010} \psi_{110} \psi_{001},$$

$$d_3 = \psi_{000} \psi_{110} \psi_{101} \psi_{011} + \psi_{111} \psi_{001} \psi_{010} \psi_{100}.$$

- ▶ **For the GHZ state** $|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$: $\tau^{(3)} = 1, \tau = 0$
and $\tau^{(3)} = 0$ for a factorized state.

State of the W -class $|\psi\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$, $\tau^{(3)} = 0, \tau = 1$.

⁷Coffman V, Kundu J and Wootters W K 2000 *Phys. Rev. A* **61** 052306. 

Refinement of the classification from **local unitary equivalence** (the LU group) ⁸

$$|B\rangle = \frac{1}{2}(|000\rangle + |101\rangle + |110\rangle + |111\rangle)$$

The three-tangle is $\tau^{(3)} = \frac{1}{4}$ and the density matrices

$$\rho_{BC} = \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \quad \rho_{AB} = \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}, \quad \rho_{AC} = \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}.$$

- ▶ Eigenvalues $\left\{ \frac{1}{16}(3 + 2\sqrt{2}), \frac{1}{16}(3 - 2\sqrt{2}), 0, 0 \right\}$ uniform over the subsystems. **For all B -states** $\tau^{(3)} = \frac{1}{4}$, $\tau_{AB} = \tau_{AC} = \tau_{BC} = \frac{1}{4}$.

⁸Acin A et al 2000 Phys. Rev. Lett. **85**, 1560

- ▶ A group W is a **Coxeter group** if it is finitely generated by a subset $S \subset W$ of involutions and pairwise relations

$$W = \langle s \in S \mid (ss')^{m_{ss'}} = 1 \rangle, \quad (1)$$

where $m_{ss} = 1$ and $m_{ss'} \in \{2, 3, \dots\} \cup \{\infty\}$ if $s \neq s'$.

The rank of the Coxeter system (W, S) is the number of gens.

- ▶ $W(F_4)$: **Symmetries of the 24-cell**

$$x_1 \text{ --- } x_2 \text{ ---}_4 x_3 \text{ --- } x_4,$$

i.e. the relations $x_1^2 = x_2^2 = x_3^2 = x_4^2 = (x_2x_3)^4 = (x_1x_2)^2 = (x_3x_4)^2 = (x_1x_3)^3 = (x_1x_4)^3 = (x_2x_4)^3 = 1$.

Finite Coxeter groups

Type	Group	Order	Rank	Coxeter diagram
A_n	S_{n+1}	$(n+1)!$	n	$x_1 - x_2 \dots x_{n-1} - x_n$
B_n	$\mathbb{Z}_2^n \rtimes S_n$	$2^n n!$	n	$x_1 -_4 x_2 \dots x_{n-1} - x_n$
D_n	$\mathbb{Z}_2^{n-1} \rtimes S_n$	$2^{n-1} n!$	n	$x_1 - x_2 - \dots -_{-x_3} \dots x_{n-2} - x_{n-1}$
$I_2(p)$	Dih_p	$2p$	2	$x_1 -_p x_2$
H_3	**	120	3	$x_1 -_5 x_2 - x_3$
F_4	**	1152	4	$x_1 - x_2 -_4 x_3 - x_4$
G_4	**	1440	4	$x_1 -_5 x_2 - x_3 - x_4$
E_6	**	51840	6	$x_1 - x_2 - x_3 -_{-x_4} - x_5 - x_6$
E_7	**	2 903 040	7	$x_1 - x_2 - x_3 -_{-x_4} - x_5 - x_6 - x_7$
E_8	**	696 729 600	8	$x_1 - x_2 - x_3 -_{-x_4} - x_5 - x_6 - x_7 - x_8$


Clifford gates are group operations stabilizing Pauli operations⁹

- ▶ Action $g \in \mathcal{P}_n$ on an n -qubit state $|\psi\rangle$ is $g|\psi\rangle$, evolves as $Ug|\psi\rangle$, and can be stabilized by U such that $(UgU^\dagger)U|\psi\rangle = U|\psi\rangle$, with $UgU^\dagger \in \mathcal{P}_n$

$$\mathcal{C}_n = \left\{ U \in U(2^n) \mid U\mathcal{P}_nU^\dagger = \mathcal{P}_n \right\}.$$

- ▶ A few relationships

$$\begin{aligned} \mathcal{C}_n/\mathcal{P}_n &\cong \text{Out}(\mathcal{P}_n) \cong \mathbb{Z}_2 \times \text{Sp}(2n, 2) \\ \tilde{\mathcal{C}}_3 &\cong \mathbb{Z}_2^6 \rtimes W'(E_7), \quad \tilde{\mathcal{B}}_3 \cong \mathbb{Z}_2^6 \rtimes W'(E_6) \end{aligned}$$

⁹Clark S, Jozsa R and Linden N 2008 *Quantum Inf. Comp.* **8** 106. 

Generating Clifford groups with quantum gates

- ▶ single qubit: $|\mathcal{C}_1| = 192$

$$\mathcal{C}_1 = \mathcal{U}_9 = \langle H, P_0 \rangle, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad P_0 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

- ▶ two qubits: $|\mathcal{C}_2| = 92160$, $|\mathcal{B}_2| = 15160$

$$\mathcal{C}_2 = \langle \mathcal{C}_1 \otimes \mathcal{C}_1, \text{Cnot} \rangle, \quad \mathcal{B}_2 = \langle \mathcal{C}_1 \otimes \mathcal{C}_1, R \rangle$$

- ▶ three-qubits: $|\mathcal{C}_3| = 743\,178\,240$ and more
 \mathcal{C}_n generated with H , P_0 and Cnot (Gottesman-Knill 1997)

Splitting: extraspecial group \times orthogonal group¹⁰

$$C_n^\pm \cong E_{2n+1}^\pm \cdot \Omega^\pm(2n, 2)$$

with $\Omega^\pm(2n, 2) = O^\pm(2n, 2)'$.

► **Single qubit Clifford group dipoles**

$$C_1^+ = G(4, 4, 2) \cong Q, \quad C_1^- \cong D_4 \cdot \mathbb{Z}_3 \cong SL(2, 3).$$

- . C_1^- is a *optimal 2-dim 2-design*
- . pair (C_1^-, C_1^+) is a BN-pair

¹⁰Planat M 2009 Preprint 0904.3691 (quant-ph);

Planat M and Solé P 2008 J. Phys. A: Math. Theor. **41** 182001.

► Two-qubit dipoles

$$\begin{aligned}C_2^+ &\cong E_{32}^+ \times S_3^2 \cong W(F_4), \\C_2^- &= B'_2 \cong E_{32}^- \cdot A_5.\end{aligned}$$

A maximal subgroup of C_2^- is $\cong SL(2, 5)$ (a *optimal 2-dim 5-design*)

► Three-qubit dipoles

$$C_3^+ \cong E_{128}^+ \cdot A_8,$$

$$C_3^- = B'_3 \cong E_{128}^- \cdot W'(E_6).$$

- ▶ **The real Clifford group** (of order 2 580 480)

$$C_3^+ = \langle 1 \otimes S, S \otimes 1, 1 \otimes \text{Swap}, \text{Swap} \otimes 1 \rangle,$$

is the 2nd largest max. subgroup of $W(E_8)$ (of order 696 729 600)

$$\langle C_3^+, \text{Tof} \rangle \cong W(E_8).$$

- ▶ **Realization of $W'(E_8)$ with two W -type gates**

$$W'(E_8) \cong O^+(8, 2) \cong \langle \sigma_x \otimes S, S_3 \rangle$$

Rows of S_3 are encoded with joined eigenstates of

$$\sigma_z \otimes (\sigma_x \otimes \sigma_z, \sigma_z \otimes \sigma_x, \sigma_y \otimes \sigma_y).$$

- From the chain of subgroups

$$W'(E_8) \supset W'(E_7) \supset W'(E_6) \supset G_{648} \supset S_4 \supset A_4,$$

one arrives at $A_4 = \langle a_1, a_2 \rangle$ with B-type generators

$$a_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 & -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & -1 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & 1 & -1 & 1 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$a_2 = \frac{1}{2} \begin{pmatrix} 0 & -1 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & -1 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & -1 & 1 & 1 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

- ▶ Let G a matrix Lie group, the Lie algebra \mathfrak{g} of G is the set of all matrices X such that e^{tX} is in G , \forall real t , and

for any $X \in \mathfrak{g}$, and for $A \in G$, then $\text{Ad}_A(X) = AXA^{-1} \in \mathfrak{g}$,

i.e. *conjugation of an element of the Lie algebra by an element of the Lie group*: **the adjoint mapping** preserves the algebra ¹¹.

- ▶ The adjoint endomorphism “Ad” can be reformulated in terms of commutators by the linear map “ad” as follows

$$\text{ad}_X : \mathfrak{g} \rightarrow \mathfrak{g} \text{ defined by } \text{ad}_X(Y) = [X, Y].$$

¹¹Remind the Clifford group \mathcal{C}

for any $X \in \mathcal{P}$, and for $A \in \mathcal{C} \subset U(n, \mathbb{C})$, then $AXA^{-1} \in \mathcal{P}$.

Lie groups/algebras are a *smooth* formulation of quantum error correction.

- ▶ **Structure constants** $[X_i, X_j] = \sum_{k=1}^n c_{ij}^k X_k$,
- ▶ **Killing form**

$$\text{Kil}(X, Z) = \text{trace}(ad(X)ad(Z)), \quad \text{Kil}^{ij} = \frac{1}{I_{\text{ad}}} c_{im}^n c_{jn}^m.$$

Non-degeneracy implies a semi-simple Lie algebra.

- ▶ **Real forms:** a real Lie algebra \mathfrak{g}_0 whose complexification is complex Lie algebra \mathfrak{g} .

Signature $(s_1, s_2) \equiv \#$ pos./neg. entries in the diagonal form of Kil;
e.g. $sl(2, \mathbb{C})$ has the real forms $sl(2, \mathbb{R}) \equiv su(1, 1)$ [non compact, of signature $(2, 1)$] and $su(2)$ [compact, of signature $(0, 3)$],
and $sl(3, \mathbb{C})$ has three real forms $sl(3, \mathbb{R})$, $su(3)$ and $su(2, 1)$.

Cartan subalgebra of \mathfrak{g} : a complex subspace \mathfrak{h} of \mathfrak{g}

- (i) For all H_1 and $H_2 \in \mathfrak{h}$, $[H_1, H_2] = 0$,
- (ii) For all $X \in \mathfrak{g}$, if $[H, X] = 0$ for all $H \in \mathfrak{h}$, then $X \in \mathfrak{h}$,
- (iii) For all $H \in \mathfrak{h}$, ad_H is diagonalizable

Root of \mathfrak{g} : a nonzero linear functional α on \mathfrak{h} such that there exists a nonzero element X of \mathfrak{g} with

$$[H, X] = \alpha(H)X, \quad \forall H \in \mathfrak{h}.$$

If α is a root, then the **root space** \mathfrak{g}_α is the space of all X in \mathfrak{g} for which $[H, X] = \alpha(H)X$ for all H in \mathfrak{h} .

Positive simple roots: linear combinations of the α_i ($i = 1..k$).

Lie groups and algebras: $sl(3, \mathbb{C})$

$$x_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, x_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, x_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, y_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$h_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, h_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

Commutation table: Cartan subalgebra $\mathfrak{h} = (h_1, h_2)$,

Simple positive roots: $\alpha_1 = (2, -1)$, $\alpha_2 = (-1, 2)$, $\alpha_3 = (1, 1)$

$[\cdot, \cdot]$	x_1	x_2	x_3	y_1	y_2	y_3	h_1	h_2
x_1	.	$-x_3$.	h_1	.	y_2	$-2x_1$	x_1
x_2		.	.	.	h_2	$-y_1$	x_2	$-2x_2$
x_3			.	x_2	$-x_1$	$h_1 + h_2$	$-x_3$	$-x_3$
y_1				.	y_3	.	$2y_1$	$-y_1$
y_2					.	.	$-y_2$	$2y_2$
y_3						.	y_3	y_3
h_1							.	.
h_2							.	.

Lie algebra attached to S_4

- ▶ Thus, \mathfrak{g}'_{A_4} has commutation table similar to $sl(3, \mathbb{C})$, and the same (positive) roots $(2, -1)$, $(-1, 2)$ and $(1, 1)$.
- ▶ Going upstream in the group sequence one arrives at a three-qubit realization of the symmetric group S_4 . The corresponding Lie algebra, of dimension 13, may be decomposed as a direct sum of simple Lie algebra

$$\mathfrak{g}_{S_4} = sl(3, \mathbb{C}) \oplus su(1, 1) \oplus u(1) \oplus u(1),$$

in which the Lie algebra $sl(3, \mathbb{C}) \oplus u(1)$ is embedded.

- ▶ Cristallography of $W(E_8)$ related to three-qubit entanglement.
- ▶ A_4 condensation relates to B -states and $sl(3, \mathbb{C})$.
- ▶ Application to **particle (and gravitation) physics** ¹².
Application to the **genetic code** ¹³: codons as three-qubit systems.

¹²Carruthers P A 1966 *Introduction to unitary symmetry* (John Wiley, N.Y.)

¹³Hornos J E M 1993 *Phys. Rev. Lett.* **71** 4401.