

Experiments with Hilbert series computations and the Anick conjecture

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In this talk I restrict my attention to graded quadratic algebras.

Gromov's rule: "nothing non-trivial could be said about arbitrary finitely-generated group"

Definition

Say that $A = k\langle x_1, \dots, x_n \mid f_i \rangle$, over an arbitrary field k , is a **quadratic algebra** generated in degree one if:

$f_i \in \mathcal{F}_2$, with respect to usual grading of the free algebra

$$\mathcal{F} = k\langle x_1, \dots, x_n \rangle = \bigoplus_{i \in \mathbb{Z}} \mathcal{F}_i$$

where $\deg x_i = 1$

and $\mathcal{F}_i = \langle \text{monomials of degree } i \rangle_k$

We are interested in how *generic* quadratic algebra behave.

What is its *Hilbert series*?

Definition

Since relations $f_i \in \mathcal{F}_2$ are homogeneous, A inherits a grading from \mathcal{F} . So, $A = \bigoplus_{i \in \mathbb{Z}} A_i$.

Associate a polynomial generating function to the sequence of dimensions $a_n = \dim A_i$:

$$H_A = \sum_{n=0}^{\infty} a_n t^n.$$

It is called a **Hilbert series** of A and is an element of $k[[t]]$.

We will answer the question on what is
a *minimal* (generic) *Hilbert series*
in the class of algebras with n generators
and d relations.

It is known due to the prominent work of E.S.Golod under supervision of his advisor I.R.Shafarevich that for an arbitrary graded algebra

$$A = k\langle x_1, \dots, x_n \mid f_1, f_2, \dots \rangle,$$

where f_1, f_2, \dots are relations,

of degrees $n_1 \leq n_2 \leq \dots$,

and r_i is a number of relations of degree i

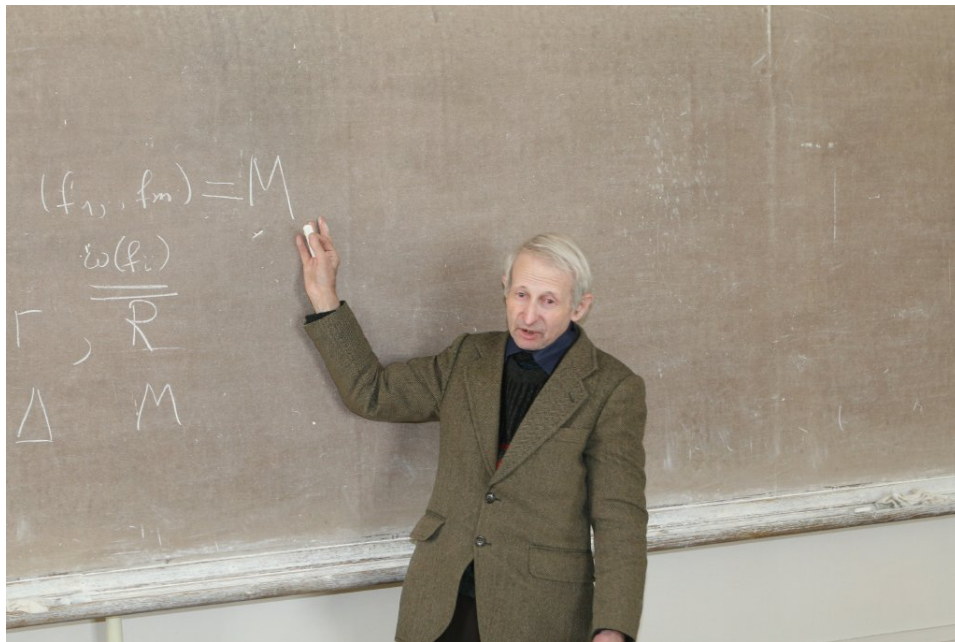
the following coefficient-wise inequality holds:

$$\left| \left(1 - nt + \sum_{i=2}^{\infty} r_i t^i \right)^{-1} \right| \leq H_A,$$

here $|P(t)|$ stands for the series where all terms of $P(t)$, starting from the first negative one are omitted.

In particular, if coefficients of the series $\left(1 - nt + \sum_{i=2}^{\infty} r_i t^i \right)^{-1}$ are positive, then the algebra is *infinite-dimensional*.

As it is well-known this estimate were used to construct Golod-Shafarevich counterexample to the *Kurosh problem* on whether any finitely generated nil algebra is nilpotent.



As it is written in the Herstein book "Non-commutative rings" (in 1972), where he explains this example:

" ... one can construct a nil but not locally nilpotent algebra thereby giving a negative answer to *Kurosh problem*; one can construct a torsion group which is not locally finite thereby settling in the negative *General Burnside Problem*. In addition the theorem gives rise to important results and examples in the theory of *p-groups* and in *class field theory*.

However it seems likely that these successful uses of the method are merely beginning, that a host of results await the application of the technique.”

It was already confirmed by the work of Agata Smoktunowicz, related to *Köthe conjecture*, where some arguments started with something very similar to Golod’s construction, and then goes further and deeper.

So, for quadratic algebras we have the Golod-Shafarevich estimate:

$$|(1 - nt + dt^2)^{-1}| \leq H_A,$$

where d is a number of quadratic relations.

In his 1983 paper "*Generic algebras and CW-complexes*", Princeton Univ. Press, *D.Anick* studied the behavior of Hilbert series of algebras given by relations and conjectured that

Conjecture (Anick) For quadratic algebras the Golod-Shafarevich estimate is attained for a generic algebra A .

Generic is understood in the sense of Zariski topology.

It is clear that the *component-wise minimal* series is in general position, at least if it is finite.

In case minimal series is infinite, it is generic in sense of Lebesgue measure over \mathbb{R} , as we mention in [Iyudu, Cameron, JSC, 2007]

Denote the *component-wise minimal series* and its coefficients by

$$H_{n,d}^{min}(t) = \sum h_q(k, n, d) t^q$$

What was known about the conjecture?

- There are examples of non-quadratic algebras, where G-Sh estimate is not attained.
- In the above paper Anick prove that for $d \geq \frac{n^2}{2}$ the conjecture is true.

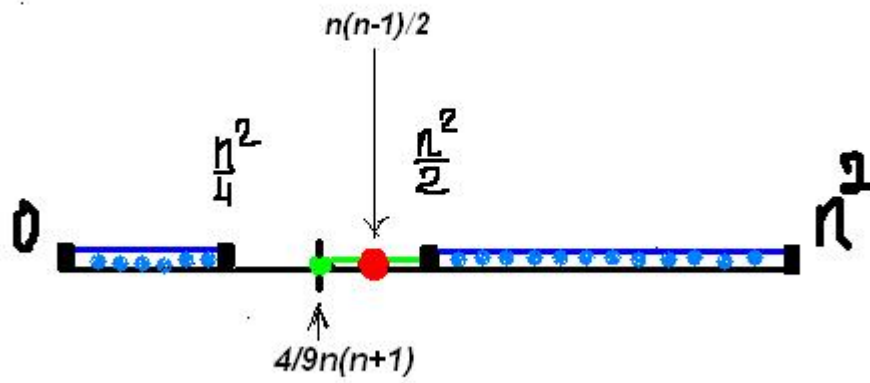
He showed that $\lfloor (n^2 + 1)/2 \rfloor$ quadratic relations can be chosen in such a way that $A_3 = 0$.

- The Anick Conjecture is also true if

$$d \leq \frac{n^2}{4}.$$

The set of relations is called *combinatorially free*, if there are no *obstructions* in the set of highest words with respect to some order: never beginning of one word coincide with the end of another.

So algebra $k\langle x_1, \dots, x_k, y_1, \dots, y_k \mid x_i y_j \rangle$, with $\frac{n^2}{4}$ relations has a minimal Hilbert series.



A special point in this picture is $d = \frac{n(n-1)}{2}$. For instance, it is the number of relations defining the algebra of commutative polynomials or any PBW algebra.

This point addressed in [Vershik, *Selecta Math. Soviet*, 1992]

Conjecture (Vershik). A generic quadratic algebra with n generators and $d = \frac{n(n-1)}{2}$ relations is finite dimensional.

In a recent preprint [Iyudu, Shkarin, 2009] we have moved this frame, which remained stable for about 25 years and also proved Vershik's Conjecture.

Theorem. *For any infinite field, the Golod–Shafarevich estimate is attained for a generic quadratic algebra with n generators and $d \geq \frac{4(n^2+n)}{9}$ quadratic relations.*

Namely, the Hilbert series of the generic algebra is:

$$H(t) = |(1 - nt + dt^2)^{-1}| =$$
$$1 + nt + (n^2 - d)t^2 + (n^3 - 2nd)t^3.$$

About the proof.

Denote

$$d(n, q) = \min\{d : H_q^{\min}(n, d) = 0\}.$$

That is, $d(n, q)$ is the minimal number of quadratic relations which guarantee vanishing of q^{th} component A_q of a generic algebra A .

More precisely, we prove:

Lemma.

$$d(n, 4) \leq \begin{cases} \frac{4n^2}{9} & \text{if } n = 3k, \\ \frac{4n^2+2n-2}{9} & \text{if } n = 3k + 2, \\ \frac{4n^2+4n-8}{9} & \text{if } n = 3k + 1. \end{cases}$$

In any case $d(n, 4) < \frac{4(n^2+n)}{9}$ for any field.

Ingredients of the proof:

Lemma. $H_q^{min}(kn, k^2d) \leq k^q H_q^{min}(n, d)$.

and

Specific examples for small number of generators.

For instance,

$$A = k\langle x_1, x_2, x_3 \mid x_1x_2, x_1x_3, x_2x_3, x_1^2 + x_2^2 + x_3^2 \rangle$$

has the Hilbert series $1 + 3t + 5t^2 + 4t^3$.

In particular, $H_4^{min}(3, 4) = 0$ for any field \mathbb{K} .

This example could be calculated directly by hands.

(and hence over a field of any characteristic!)

We also used examples of algebras with $(n, d) = (4, 6), (4, 5), (7, 19)$ which look like this:

Let $R = \mathbb{Z}_2\langle x_1, x_2, x_3, x_4 \rangle / I$ with $I = \text{Id}\{f_1, \dots, f_6\}$ and

$$\begin{aligned} f_1 &= x_1x_2, & f_2 &= x_1x_4 + x_4x_2 + x_2x_3, \\ f_3 &= x_1x_3 + x_3x_4 + x_4x_1, & f_4 &= x_1^2 + x_2^2 + x_3^2 + x_4^2, \\ f_5 &= x_3x_4 + x_4x_2 + x_2x_4, & f_6 &= x_2x_3 + x_3x_1 + x_1x_3. \end{aligned}$$

Then the Hilbert series of R is $H_R(t) = 1 + 4t + 10t^2 + 16t^3 + t^4 = |(1 - 4t + 6t^2)^{-1}|$.

Let $R = \mathbb{Z}_2\langle x_1, x_2, x_3, x_4 \rangle / I$ with $I = \text{Id}\{f_1, \dots, f_5\}$
and

$$\begin{aligned} f_1 &= x_1^2 + x_2^2 + x_3^2 + x_4^2, & f_2 &= x_1x_2 + x_2x_3 + x_3x_4, \\ f_3 &= x_4x_1 + x_1x_3 + x_3x_2, & f_4 &= x_1x_3 + x_3x_2 + x_2x_4, \\ f_5 &= x_1x_4 + x_4x_3 + x_3x_2 + x_2x_4. \end{aligned}$$

Then the Hilbert series of R is $H_R(t) = 1 + 4t + 11t^2 + 24t^3 + 41t^4 + 44t^5 = |(1 - 4t + 5t^2)^{-1}|$.

Let $R = \mathbb{Z}_2\langle x_1, \dots, x_7 \rangle / I$ with $I = \text{Id}\{f_1, \dots, f_{19}\}$
and

$$f_1 = x_1x_7,$$

$$f_2 = x_3x_7 + x_4x_6 + x_6x_2,$$

$$f_3 = x_5x_7 + x_6x_4 + x_3x_5 + x_2x_1 + x_4x_3,$$

$$f_4 = x_7x_1 + x_1x_6,$$

$$f_5 = x_7x_2 + x_6x_1 + x_1x_5,$$

$$f_6 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2,$$

$$f_7 = x_2x_7 + x_7x_3,$$

$$f_8 = x_6x_7 + x_3x_6 + x_4x_5 + x_5x_2,$$

$$f_9 = x_7x_5 + x_2x_6 + x_5x_3 + x_1x_4 + x_3x_2,$$

$$f_{10} = x_5x_7 + x_7x_6,$$

$$f_{11} = x_7x_6 + x_6x_2 + x_5x_1 + x_3x_4,$$

$$f_{12} = x_7x_4 + x_6x_3 + x_2x_5 + x_3x_2 + x_4x_1,$$

$$f_{13} = x_7x_5 + x_4x_7,$$

$$f_{14} = x_7x_2 + x_3x_6 + x_6x_4,$$

$$f_{15} = x_2x_7 + x_6x_5 + x_5x_4 + x_3x_1 + x_4x_2,$$

$$f_{16} = x_3x_7 + x_7x_4,$$

$$f_{17} = x_4x_7 + x_2x_6 + x_6x_3,$$

$$f_{18} = x_7x_3 + x_4x_6 + x_5x_2 + x_2x_4 + x_3x_1,$$

$$f_{19} = x_6x_7 + x_6x_4 + x_2x_6 + x_2x_5 + x_3x_5 + x_4x_5.$$

Then the Hilbert series of R is $H_R(t) = 1 + 7t + 30t^2 + 77t^3 = |(1 - 7t + 19t^2)^{-1}|$.

In these examples computations were done using the bunch of programs GRAAL (Graded Algebras) specially designed for Gröbner bases and Hilbert series computations in graded algebras.

Authors: A.Verevkin, Uljanovsk brunch of Moscow State University and A.Kondratyev, RISC Linz.

Each time we get a minimal series over the field of characteristic p , for a specific p . Hence it is also minimal over the field of char. 0.

Another approach to producing examples of algebras with minimal series.

In the above mentioned paper [*Iyudu, Cameron, JSC, 2007*] we confirmed the conjecture for $n \leq 7$ and $d = n(n - 1)/2$.

The idea we used based on the fact that minimal series should be reached for a generic algebra, so we tried to mimic *generic coefficients*. And this is an example of how they look like:

Example The algebra A over the field $k = \mathbb{Z}_{17}$ given by the relations $A = k\langle a, b, c \rangle / I$, $I =$

$$\begin{cases} ac + 2ba + 9b^2 + 3ca + 9cb + 8c^2, \\ 3ab + 5ac + 7ba + b^2 + 8bc + 4ca + cb + 2c^2, \\ 10a^2 + 2ab + 11ac + 2ba + 8b^2 + 4bc + 9ca + 7cb + 5c^2 \end{cases}$$

has the Hilbert series

$$H_A = 1 + 3t + 6t^2 + 9t^3 + 9t^4 = |(1 - 3t + 3t^2)^{-1}|.$$

Corollary. *For any infinite field \mathbb{K} , $n \geq 17$ and $d = \frac{n(n-1)}{2}$, generic algebra has Hilbert series $H(t) = |(1 - nt + dt^2)^{-1}| = 1 + nt + \frac{n(n+1)}{2}t^2 + n^2t^3$.*

Remark. We can prove it for $n \geq 5$, but for that should consider field of characteristic 0.

Solution of the Vershik question:

Theorem. *Over a field of characteristic 0 for $d = \frac{n(n-1)}{2}$ a generic quadratic algebra A has the Hilbert series*

$$H_A(t) = \begin{cases} 1 + nt + \frac{n(n+1)}{2}t^2 + n^2t^3 & \text{if } n \geq 5; \\ 1 + 4t + 10t^2 + 16t^3 + t^4 & \text{if } n = 4; \\ 1 + 3t + 6t^2 + 9t^3 + 9t^4 & \text{if } n = 3, \end{cases}$$

In particular, A is finite dimensional.

Our technique also allows to get some asymptotic results.

Proposition. *There exists the limit*

$$\lim_{n \rightarrow \infty} \frac{d(n, q)}{n^2} = \alpha(q) = \inf \left\{ \frac{d(n, q)}{n^2} : n \in \mathbb{N} \right\}.$$

We call the number $\alpha(q)$ the *asymptotic border*.

If we suppose that Anick's conjecture is true, then we should have: $\alpha(3) = \frac{1}{2}$, $\alpha(4) = \frac{3-\sqrt{5}}{2}$, $\alpha(5) = \frac{1}{3}$, \dots $\lim_{q \rightarrow \infty} \alpha(q) = \frac{1}{4}$.

Theorem.

$$\alpha(2) = 1 \text{ (trivial),}$$

$$\alpha(3) = \frac{1}{2} \text{ (Anick),}$$

$$\alpha(4) = \frac{3 - \sqrt{5}}{2} = 1 - \textit{golden section},$$

$$\alpha(5) = \frac{1}{3},$$

$$\alpha(6) \leq \frac{5}{16}.$$