Strongly regular graphs with no triangles: hidden history, challenges, computer algebra results

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This talk is based on cooperative work of M. Klin, A. Woldar and MZ.

Motivation

- The strongly regular graph (SRG) with parameters $(100, 22, 0, 6)$ is known as the Higman-Sims graph since it was discovered by Higman and Sims in 1968 and used for describing the sporadic simple Higman-Sims group.
- It is less known that Dale Mesner discovered the same graph more than a decade prior to Higman and Sims.
- In his Ph.D. thesis in 1956 Mesner constructed the graph, and proved that only four possible SRGs with that parameters are possible.
- Mesner constructed the adjacency matrix as a block matrix for one of the possibilities, but did not look into the three other options.
- In 1964, in mimeographed notes, he proved uniqueness of the graph.

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   - Triangle free SRGs
   - Known tfSRGs
2. SRGs with no triangles via Dale Mesner
   - SRGs with no triangles
   - Negative Latin SRG
3. $NL_2(10)$: construction by Mesner
   - Introduction
   - The nicest biplane on 16 points
   - Anatomy of the design D
   - Our aim - non-edge decomposition of $\Gamma = NL_2(10)$
   - Subsequent results
4. Research directions
   - Known tfSRGs and beyond
   - Graham Higman's ideas
   - Small tfSRGs inside $NL_2(10)$

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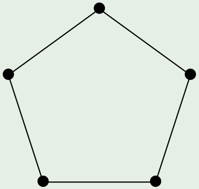
- If we denote the adjacency matrix of a graph by A, then an algebraic formulation of the condition is:

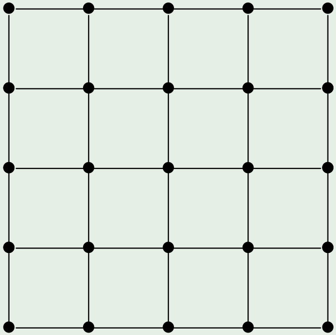
$$A^2 = kI + \lambda A + \mu(J - I - A)$$

Example (Pentagon)



$SRG(5, 2, 0, 1)$

Example (Lattice graph)



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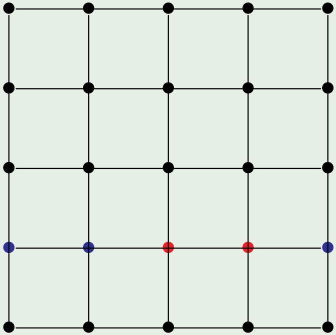

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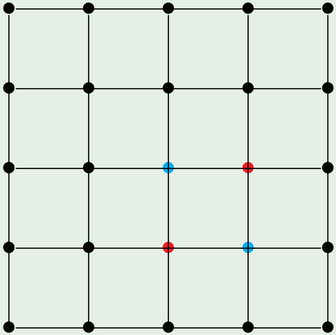

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Some facts about SRGs

- The complement graph of an SRG with parameters (v, k, λ, μ) is an SRG with parameters $(v, v-k-1.v-2-2k+\mu, v-2k+\lambda)$.

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- Disconnected SRG: $\mu = 0$, $n \circ K_m$.
- SRG is primitive if both it and its complement connected.
- Therefore an SRG is imprimitive if $\mu = 0$ or $\mu = k$.

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- $(v - k - 1)\mu = k(k - \lambda - 1)$

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- An SRG has exactly three eigenvalues, k of multiplicity 1 and $r = \frac{1}{2}\left[(\lambda-\mu)+\sqrt{(\lambda-\mu)^2+4(k-\mu)}\right]$, $s = \frac{1}{2}\left[(\lambda-\mu)-\sqrt{(\lambda-\mu)^2+4(k-\mu)}\right]$, whose multiplicities:
- $f = \frac{1}{2}\left[(v-1)-\frac{2k+(v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}\right]$ and $g = \frac{1}{2}\left[(v-1)+\frac{2k+(v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}\right]$ are integers.

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- Parameters that satisfy these conditions are called feasible parameters.
- A table of feasible parameters up to $v = 1300$ is at http://www.win.tue.nl/~aeb/graphs/srg/srgtab.html

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- The algebraic constraints:
  $2k = \mu(v-1)$ or $\mu^2 + 4(k-\mu)$ is a square.

Moore graphs, $\mu = 1$

- A special case of tfSRGs is $\mu = 1$.
- That is, graphs with no triangles or quadrangles.
- These graphs are called Moore graphs.
- Possible only for $k = 2, 3, 7, 57$. $v = k^2 + 1$.
- There exists unique graph for $k = 2, 3, 7$.
- Existence of Moore graph with $k = 57$ is one of main open questions in graph theory.

Feasible parameters for tfSRGs

9 parameter sets with $n \leq 100$:

No.	v	k	λ	μ	Existence
1	5	2	0	1	Yes, pentagon
3	10	3	0	1	Yes, Petersen
16	16	5	0	2	Yes, Clebsch
15	28	9	0	4	No, 1956
34	50	7	0	1	?, Hoffman-Singleton
39	56	10	0	2	?, Gewirtz
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- This table was first compiled by Dale Mesner.
- The No. column corresponds to a table of feasible parameters for general SRGs.
- the Existence column represents what was known to Mesner or proven by him.

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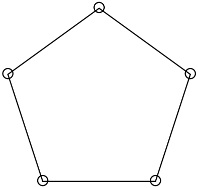
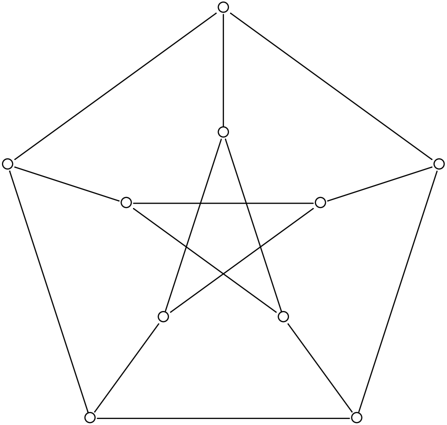
$(5, 2, 0, 1)$ Pentagon



$(10, 3, 0, 1)$ Petersen graph



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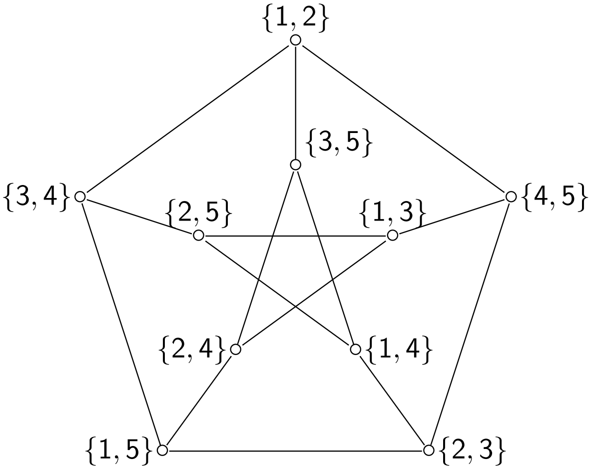

- Pentagon is a subgraph of Petersen graph.

$(16, 5, 0, 2)$ Clebsch graph

- Clebsch graph $\square_5$ — Cayley graph $Cay(E_{16}, \{0001, 0010, 0100, 1000, 1111\})$.
- It is $NL_1(4)$.
- Q_4 plus diagonals.
- $Aut(\square_5) \cong E_{2^4} \rtimes S_5$.
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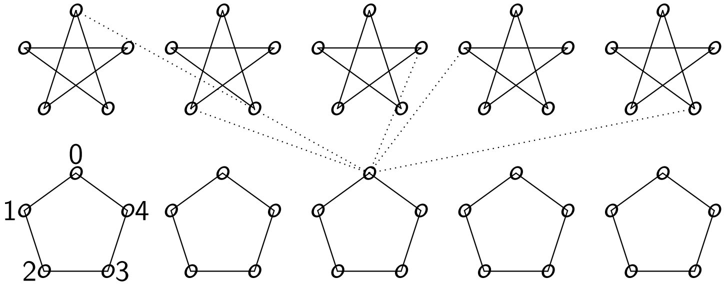
- $Aut(\square_5) \cong E_{2^4} \rtimes S_5$.
- Clebsch graph is $SRG(16, 5, 0, 2)$.
- Petersen graph is graph of non-neighbours of a vertex in Clebsch graph.

$(50, 7, 0, 1)$ Hoffman-Singleton graph



- Construction: Five Pentagons and 5 Pentagrams. Vertex i of Pentagon j is adjacent to vertex $hj + i$ of Pentagram h.

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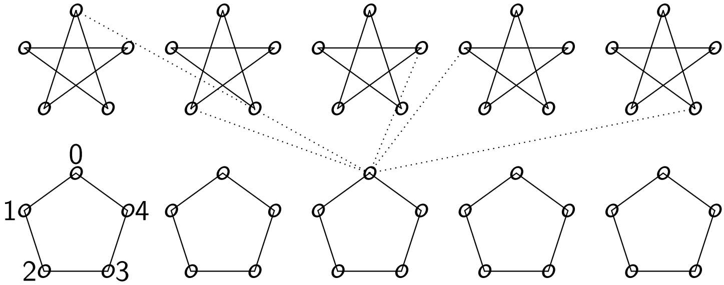

- Construction: Five Pentagons and 5 Pentagrams. Vertex i of Pentagon j is adjacent to vertex $hj + i$ of Pentagram h.
- Petersen graph is a subgraph of Hoffman-Singleton graph. Clebsch graph is not.

$(56, 10, 0, 2)$ Sims-Gewirtz graph

- Hoffman-Singleton graph is not a subgraph of Sims-Gewirtz graph. Clebsch graph is not a subgraph.

$(77, 16, 0, 4)$

- Hoffman-Singleton graph is not a subgraph of $SRG(77, 16, 0, 4)$. Sims-Gewirtz graph is a subgraph.

$(100, 22, 0, 6)$ $NL_2(10)$

This graph is the main topic of the next part of the talk.

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SRGs with no triangles via Mesner

- Let Γ be a SRG with no triangles, $\Gamma = (V, E)$.
- Let $x \in V$. We denote by $\Gamma(x)$ and $\Delta(x)$ the sets of neighbours and non-neighbours of x.
- The bipartite subgraph of Γ with two parts $\Gamma(x)$ and $\Delta(x)$ as incidence graph of a suitable incidence structure C.

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Theorem (2.6, DM, 1956)

(i) *C is a BIBD with the parameters*
$\hat{v} = k$
$\hat{b} = l = n - k - 1$
$\hat{r} = k - 1$
$\hat{k} = \mu$
$\hat{\lambda} = \mu - 1$
(caret ^ denotes parameters of BIBD);

(ii) *Any block of C is disjoint from at least $k - \mu$ blocks.*

Lists of feasible parameters

- Mesner created a list of feasible parameters of SRGs with $v \leq 100$.
- He collected all known information about existence of SRGs, presenting new constructions.
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Non existence of tfSRGS on 28,64 vertices

Result (p.61, 1956)
The graphs with the numbers 15 and 50 are impossible.
The proof is based on the variance counting, classical tool in statistics.

Remark Modern way exploits Krein inequalities, which rely on spectral graph theory.

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Negative Latin Square graphs

- DM considered already in 1956 SRGs of negative Latin square type, that is SRGs with the parameters:
- $v = n^2$
- $k = g(n+1)$
- $\lambda = (g+1)(g+2) - n - 2$
- $\mu = g(g+1)$
- A graph with these parameters is denoted by $NL_g(n)$.
- The name comes from changing the sign in the parameters of a graph arising from a set of k MOLS.

Negative Latin Square graphs with no triangles

- Letting in addition $\lambda = 0$, DM gets

$$n = (g+1)(g+2) - 2 = g^2 + 3g = g(g+3)$$

- Thus we obtain a one-parameter series of putative parameters for tfSRGs of negative Latin square type, $NL_g(g^2 + 3g)$:
- $n = g(g+3)$
- $v = g^2(g+3)^2$
- $k = g(g^2 + 3g + 1)$
- $\lambda = 0$
- $\mu = g(g+1)$.

Corresponding design

- The parameters of the corresponding design C are
- $\hat{v} = g(g^2 + 3g + 1)$,
- $\hat{b} = (g^2 + 2g - 1)(g^2 + 3g - 1)$,
- $\hat{r} = (g + 1)(g^2 + 2g - 1)$,
- $\hat{k} = g(g + 1)$,
- $\hat{\lambda} = g^2 + g - 1$.

Structure of the design

Initial elements of theory of such tfSRGs appear already in 1956.

More transparent and general approach is presented in 1964.

Proposition (Equality 18.5, 1964)

Each block is disjoint from at least $g^2(g+2)$ other blocks.

Proposition (Lemma 8.2, 1964)

Each block is disjoint from exactly $g^2(g+2)$ blocks and intersects each remaining block in exactly g elements.

(In modern terms: design C is quasisymmetric)

Non-neighbours subgraph

Theorem (Theorem 8.3, 1964)

The subgraph of non-neighbours of a vertex x in $NL_g(g^2+3g)$ is a tfSRG with the parameters
$\tilde{v} = (g^2+2g-1)(g^2+3g+1)$,
$\tilde{k} = g^2(g+2)$,
$\tilde{\lambda} = 0$,
$\tilde{\mu} = g^2$.

Corollary (Corollary 8.4.1, 1964)

The existence of BIBD C with the properties as above implies existence of $NL_g(g^2+3g)$.

Further designs

Theorem (Theorem 8.5, 1964)

Existence of $NL_g(g^2 + 3g)$ implies the existence of 4 substructures which are BIBDs, namely:

- *symmetric BIBD with $g^2(g+2)$ points,*
- *BIBD with $g(g+1)$ points and $(g+1)(g+2)(g^2+g-1)$ blocks,*
- *two BIBDs with $g^2(g+2)$ points and $(g+1)(g+2)(g^2+g-1)$ blocks.*

Later results

Observation (1964)

Uniqueness of design C with the required properties implies uniqueness of $NL_g(g^2+3g)$.

Remark. Modern interpretation of results of DM includes the intersection diagram for the non-edge model of $NL_g(g^2+3g)$.

Example (Clebsch graph, revisited)

$$g = 1$$

$$\hat{v} = 5$$

$$\hat{b} = 10$$

$$\hat{k} = 2$$

$$\hat{r} = 4$$

$$\hat{\lambda} = 1$$

This is trivial (complete) BIBD which is unique.

$$\Downarrow$$

Uniqueness of the Clebsch graph.

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Construction of $NL_2(10)$ — SRG-$(100, 22, 0, 6)$

- In order to construct $\Gamma = NL_2(10)$ it is necessary to start from a quasisymmetric design C with 22 points and 77 blocks of size 6.
- An example of such design was constructed by DM in 1956. He proved that up to isomorphism there are at most 4 such designs.
- In 1964 DM proved the uniqueness of design C with requested properties.
- This implies uniqueness of graph Γ.

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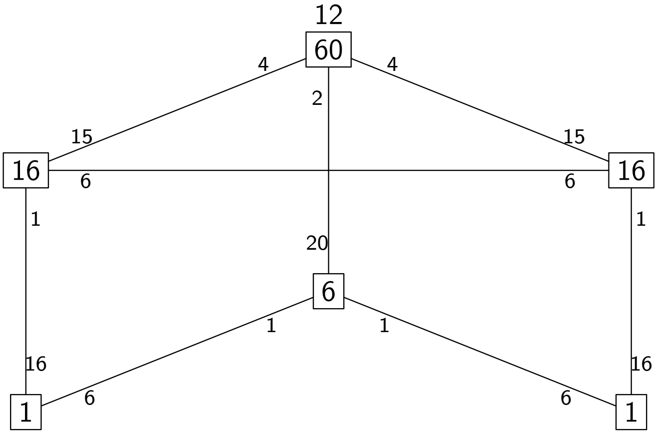
- In 1964 DM proved the uniqueness of design C with requested properties.
- This implies uniqueness of graph Γ.
- Mesner's presentation of the adjacency matrix corresponds to association schemes and equitable partitions in modern algebraic graph theory nomenclature.
- We want to keep the spirit of Mesner's presentation, but not necessarily the language.
- We use a computer for calculation - GAP (with GRAPE and nauty) and COCO.

Intersection diagrams

- An intersection diagram corresponds to an equitable partition of the graph.
- The number inside each block is the number of vertices in this part.
- The number near block A on the line from block A to block B, denotes the number of neighbors that each vertex from block A has in block B.
- The number on a loop is the number of neighbors a vertex has in the same block.
- No line between two blocks means that there are no edges between vertices of the corresponding parts of the graph.

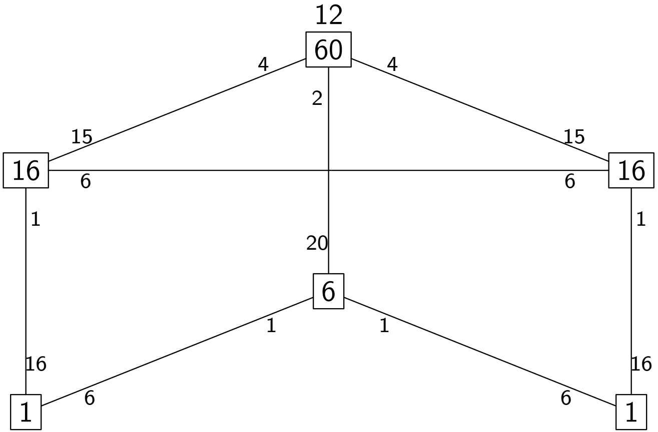
Intersection diagram of $NL_2(10)$

In the language of equitable partitions, the block form of the adjacency matrix that was presented by Mesner corresponds to the following intersection diagram, which we call non-edge decomposition.



Intersection diagram of $NL_2(10)$

What we want is to realize the diagram - to define which are the vertices in each part, and describe the adjacency in a way that will fit the diagram.



Biplanes

- A biplane is a symmetric BIBD with $\lambda = 2$.
- In other words, an incidence structure such that:
  - Every block has k points.
  - Every point is in k blocks.
  - Every two points are in exactly two common blocks.
- A biplane with parameter k has $\binom{k}{2} + 1$ points (and blocks).

Biplanes on 16 points

- For $k = 6$, biplane has 16 points and 16 blocks.
- There are three such biplanes.
- All are self dual.
- Automorphism groups of orders 384, 768 and 11520.
- The automorphism group of of order $11520 = 16 \cdot 15 \cdot 48$ is 2-transitive on points.
- This is the highest symmetry such a biplane can have.
- Therefore we consider this biplane as the nicest biplane on 16 points.

Nicest biplane on 16 points

- We construct an incidence structure $D = (\mathcal{P}, \mathcal{B})$.
- The set of points $\mathcal{P}$ is the set of vertices of Clebsch graph. $|\mathcal{P}| = 16$.
- For each vertex, a block is the set including the vertex and its 5 neighbors.
- The set of blocks $\mathcal{B}$ is also of size 16.
- D is a biplane.

Blocks of D

$\{0, 1, 2, 4, 8, 15\}$
$\{0, 1, 3, 5, 9, 14\}$
$\{0, 2, 3, 6, 10, 13\}$
$\{1, 2, 3, 7, 11, 12\}$
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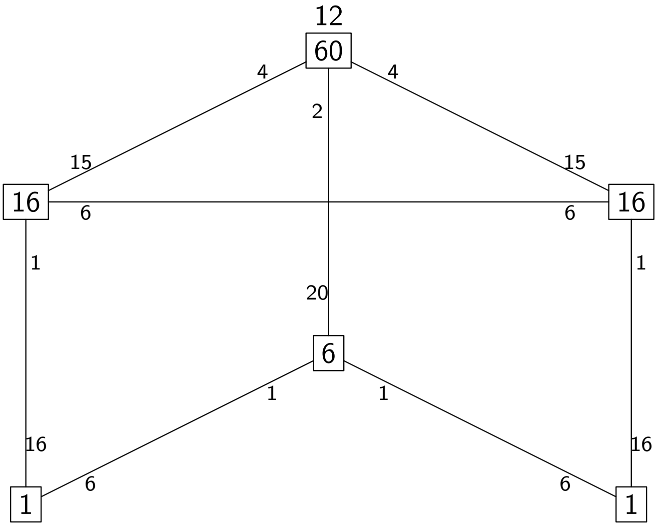
D is the nicest biplane

- $Aut(\square_5)$ acts faithfully on D.
- Therefore D is isomorphic to the nicest biplane on 16 points.
- $Aut(\square_5) \cong E_{2^4} \rtimes S_5$.
- $Aut(D) \cong E_{2^4} \rtimes S_6$.
- The change from S_5 to S_6 comes from the fact that the reverse process of constructing the graph from the design can produce six different copies of the Clebsch graph.

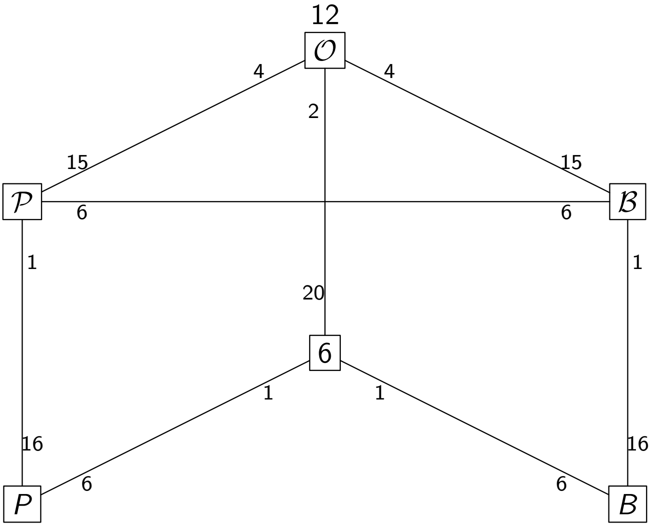
Ovals

- An oval in a design is a set of points that intersect each block in exactly 2 or 0 points..
- In a biplane on 16 points an oval has 4 points.
- in D ovals can be constructed by
  - taking an edge of $\square_5$: $\{a, b\}$.
  - Find the 6 vertices not adjacent to both a and b.
  - The induced subgraph on this set is a 1-factor.
  - Any edge of this one factor with a and b gives an oval.
- There are $\frac{16 \cdot 5 \cdot 3}{2 \cdot 2} = 60$ such ovals.
- Those are actually all ovals.
- Set of 60 ovals: $\mathcal{O}$.
- A block from $\mathcal{B}$ is disjoint to exactly 15 ovals.

Intersection diagram



Actualization of the diagram (almost)



Magical structure I

- We take 15 involutions of regular action of E_{2^4}.
- Each involution naturally partitions the set of size 16 into eight unordered pairs.
- Σ_1 is the set of those 15 partitions.

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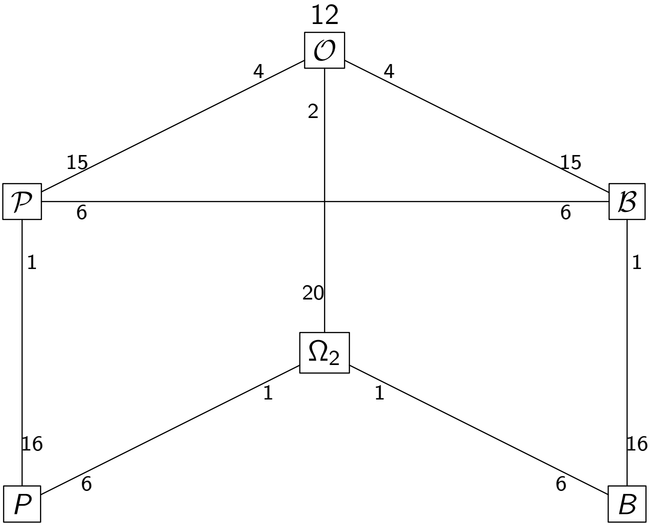
- Each involution naturally partitions the set of size 16 into eight unordered pairs.
- Σ_1 is the set of those 15 partitions.
- Consider incidence structure $(\mathcal{P}, \mathcal{O})$.
- Computation results:
  - There are 105 spreads in this structure.
  - The automorphism group has orbits of lengths 90 and 15 on the set of those spreads.
  - The orbit of length 15 is a partition of $\mathcal{O}$.
  - We denote the orbit of length 15 by Σ_2.

Magical structure II

- Incidence structure (Σ_1, Σ_2) (with incidence defined by refinement), is $GQ(2)$.
- By computer search, we find a structure M consisting of 5 elements of Σ_2:
  {{{0,1,10,11},{2,3,8,9},{4,5,14,15},{6,7,12,13}},
  {{0,3,12,15},{1,2,13,14},{4,7,8,11},{5,6,9,10}},
  {{0,2,5,7},{1,3,4,6},{8,10,13,15},{9,11,12,14}},
  {{0,4,9,13},{1,5,8,12},{2,6,11,15},{3,7,10,14}},
  {{0,6,8,14},{1,7,9,15},{2,4,10,12},{3,5,11,13}}}
- The orbit of M under the group G is of length 6.
- This orbit is denoted by Ω_2.
- Every element of Σ_2 is in exactly two elements of Ω_2, therefore
- Every element of $\mathcal{O}$ is in exactly two elements of Ω_2.

Actualization of the diagram



Adjacency in the graph

- A vertex from $\mathcal{P}$ is adjacent to 15 ovals that it is incident to.
- A vertex from $\mathcal{B}$ is adjacent to 15 ovals to which it is disjoint.
- Two ovals are adjacent if they are disjoint, and they belong to two different spreads in Σ_2.
- Thus, the graph is regular of valency 22.
- It has no triangles.
- $\mu = 6$.

Underlying design for the construction

- In fact, the design C, necessary to DM, was known for many years. This is Witt design W_{22} (in fact 3-design), which was known also to Carmichael.
- It is constructed as one-point extension of the unique projective plane of order 4.
- Witt proved (1938) uniqueness of $W_{22} = S(3, 6, 22)$ and that $Aut(W_{22}) = Aut(M_{22})$.
- Knowledge of W_{22} and its automorphism group was a starting point for Higman & Sims (1968) in the second (independent) appearance of Γ and the group $Aut(\Gamma)$.
- Recall that $Aut(\Gamma)$ contains new at that time sporadic simple group as a subgroup of index 2.

Structure of the underlying design

DM was not aware that his design C is (a quasisymmetric) Steiner system.

Lemma

For a 2-design D, any two of the following imply the third:

(i) *D is a 3-design;*

(ii) *D is a QSD with $x = 0$;*

(iii) *D has $\frac{v(v-1)}{k}$ blocks.*

The proof is known for a few decades, see e.g. Cameron, van Lint (1991).

Extensible designs

- Oppositely to DM, a common way to construct W_{22} goes via procedure of a transitive extension. This is the way used by Witt.
- It was P. Cameron who considered systematically extension of symmetric designs.
- The infinite series requested by DM is Cameron's main one-parameter infinite series of feasible parameters of extensible symmetric BIBDs with $(\lambda + 2)(\lambda^2 + 4\lambda + 2)$ points.

Motivation

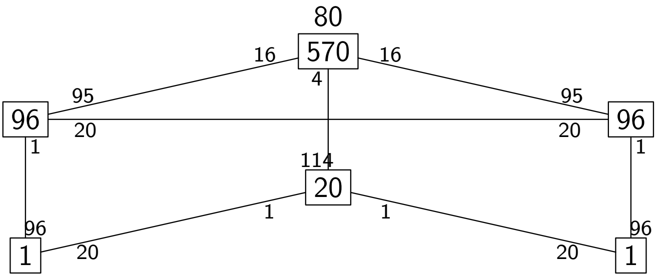
Construction of a new tfSRG remains a great challenge in modern algebraic graph theory.
First, what is coming to mind:
to go ahead with $NL_g(g^2 + 3g)$.

Case $g = 3$

- An attractive lead for a few decades was to start from a biplane on 56 points and to extend it to a 3-design on 57 points.
- 5 such biplanes were known.
- B. Bagchi twice claimed that they are not extensible (see also Cameron & van Lint, 1991).
- A fatal flaw was detected by A. E. Brouwer.
- Finally, P. Kaski and P. Ostergard proved that there exist exactly 5 biplanes on 56 points, none of them is extensible.
- (This is done via exhaustive computer search: 316 machines running in parallel for two months.)
- Thus $NL_3(18)$ does not exist!
- Independently proved by A. Gavrilyuk and A. Makhnev, without the use of a computer.

Case $g = 4$

- We wish to construct $NL_4(28) = SRG(784, 116, 0, 20)$.
- An intersection diagram for $NL_4(28)$ will look like this:



- One who wishes to imitate DM's way of non-edge decomposition has to start from symmetric BIBD on 96 points.

Case $g = 4$

- $2-(96,20,4)$ design.
- Millions of such designs are available via the approach coined by W. Wallis, late D. Fon-Der-Flaass and extended by M. Muzychuk.
- At the beginning highly symmetrical examples of such designs may be inspected.
- For example, Law, Praeger and Reichard showed that there are exactly four flag transitive $2-(96,20,4)$ designs.
- None of them has 570 ovals, so the same construction cannot be used.
- The design C has parameters $2-(116,20,19)$ and 667 blocks.

1. Strongly regular graphs
   - Basics
   - Triangle free SRGs
   - Known tfSRGs
2. SRGs with no triangles via Dale Mesner
   - SRGs with no triangles
   - Negative Latin SRG
3. $NL_2(10)$: construction by Mesner
   - Introduction
   - The nicest biplane on 16 points
   - Anatomy of the design D
   - Our aim - non-edge decomposition of $\Gamma = NL_2(10)$
   - Subsequent results
4. Research directions
   - Known tfSRGs and beyond
   - **Graham Higman's ideas**
   - Small tfSRGs inside $NL_2(10)$

Graham Higman's ideas

- The original result of G. Higman was unpublished, it appears in book of Cameron (1999).
- The goal is to provide necessary conditions for the existence of certain automorphisms of putative SRG with given parameters.
- Structure of subgraphs, induced by fixed points, is of a great significance.
- The original result of Higman was a proof of the fact that there is no vertex-transitive tfSRG of valency 57 (Moore graph of valency 57).

Limits on possible automorphisms of tfSRGs

- These techniques are nowadays exploited by followers of Higman, namely by A. Makhnev et al and recently by M. Mačaj and J. Širáň.
- Striking restrictions for the order of the automorphism group of a putative Moore graph of valency 57 are obtained by Mačaj and Širáň.
- It is a good time to start hunting for such a graph relying on those results.

SRG-$(162, 21, 0, 3)$

- The smallest open case is a tfSRG with
  $v = 162$
  $k = 21$
  $\lambda = 0$
  $\mu = 3$.
- Mačaj (extending Makhnev et al) obtained very strong restrictions on the order and structure of automorphisms of such putative graph.
- The corresponding BIBD has parameters $\hat{v} = 21$, $\hat{b} = 140$, $\hat{r} = 20$, $\hat{k} = 3$, $\hat{\lambda} = 2$.

Small tfSRGs inside $NL_2(10)$

- We are looking for smaller tfSRGs as subgraphs of $NL_2(10)$.
- In case of tfSRGs, a subgraph is an induced subgraph.
- Easy to see more generally: A subgraph of diameter two in a graph with no triangles is an induced subgraph.
- All 6 smaller tfSRGs are subgraphs of $NL_2(10)$.
- Except for the Petersen graph, all embeddings are unique up to $Aut(NL_2(10))$.

Details: Pentagon

Lemma

Let Γ be a tfSRG with the parameters (v, k, λ, μ). Then Γ contains exactly p pentagons, where

$$p = \frac{vk(k-1)(k-\mu)\mu}{10}.$$

- Proof is done by simple counting arguments of local nature.

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- Therefore there are $\frac{100 \cdot 22 \cdot 21 \cdot 16 \cdot 6}{10} = 443520$ Pentagons.
- They all belong to the same orbit under action of $Aut(NL_2(10))$.
- The stabilizer of a Pentagon in $Aut(NL_2(10))$ is of order 200.

Pentagons in other tfSRGs

name	v	k	μ	$\|Aut(\Gamma)\|$	p	$\|H\|$
Pentagon	5	2	1	10	1	10
Petersen	10	3	1	120	12	10
Clebsch	16	5	2	1920	192	10
HoSi	50	7	1	$2^5 \cdot 3^2 \cdot 5^3 \cdot 7$	1260	200
Gewirtz	56	10	2	$2^8 \cdot 3^2 \cdot 5 \cdot 7$	$2^7 \cdot 3^2 \cdot 7$	10
Mesner	77	16	4	$2^8 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	$2^7 \cdot 3^2 \cdot 7 \cdot 11$	10
$NL_2(10)$	100	22	6	$2^{10} \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	200

- In all 6 cases there is exactly one orbit of pentagons.
- For 4 cases, the stabilizer H of a pentagon is "natural", that is it coincides with D_5.
- For two cases, that is HoSi and $NL_2(10)$, the stabilizer H is "large", up to isomorphism the same group.

Details: Petersen

- There are $35,481,600$ Petersen graphs inside $NL_2(10)$.
- They are in 5 orbits.
- Stabilizers are of orders 6, 6, 24, 48, 240.

Details: Petersen

- There are 35, 481, 600 Petersen graphs inside $NL_2(10)$.
- They are in 5 orbits.
- Stabilizers are of orders 6, 6, 24, 48, 240.
- The search uses the fact that there is only one orbit of Pentagons.

Details: Clebsch

- There are 924000 Clebsch graphs inside $NL_2(10)$.
- All are in one orbit.
- Petersen graph is embedded in Clebsch graph.
- Only graphs from one of the orbits (with stabilizer of order 6) is in Clebsch within $NL_2(10)$.

Details: Hoffman-Singleton

- there are 704 Hoffman-Singleton graphs embedded in $NL_2(10)$.
- One orbit, stabilizer of order 126000.
- The stabilizer is equal to automorphisms group of Hoffman-Singleton graph.
- The induced subgraph on the other 50 vertices is also Hoffman-Singleton.
- We have 352 splittings of $NL_2(10)$ into two Hoffman-Singleton graphs.

Details: Sims-Gewirtz

- 1030 Sims-Gewirtz graphs embedded in $NL_2(10)$.
- One orbit, stabilizer of order 86040.
- The stabilizer is equal to automorphisms group of Sims-Gewirtz graph.

Details: SRG(77,16,0,4)

- 100 $SRG(77, 16, 0, 4)$ inside $NL_2(10)$.
- One orbit, stabilizer of order 887040.

Details: SRG(77,16,0,4)

- 100 $SRG(77, 16, 0, 4)$ inside $NL_2(10)$.
- One orbit, stabilizer of order 887040.
- Each of them is induced on the 77 non-neighbours of a vertex.
- It is also possible to show that those are all embeddings without help of a computer.

Algorithm used for the last three results

Let $G = Aut(NL_2(10))$, G_0 is the stabilizer of a vertex.

1. Find orbits of G_0 on subsets of size $\{7, 10, 16\}$ of 22 neighbours of 0.
2. For each of $\{5, 11, 5\}$ representatives of orbits, find other vertices which have exaclty $\{1, 2, 4\}$ neighbours in the representative.
3. In all cases only one representative has enough ($\{42, 45, 60\}$) such vertices.
4. In all cases taking those $\{42, 45, 60\}$ with 0 and the representative gives the required graph.

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