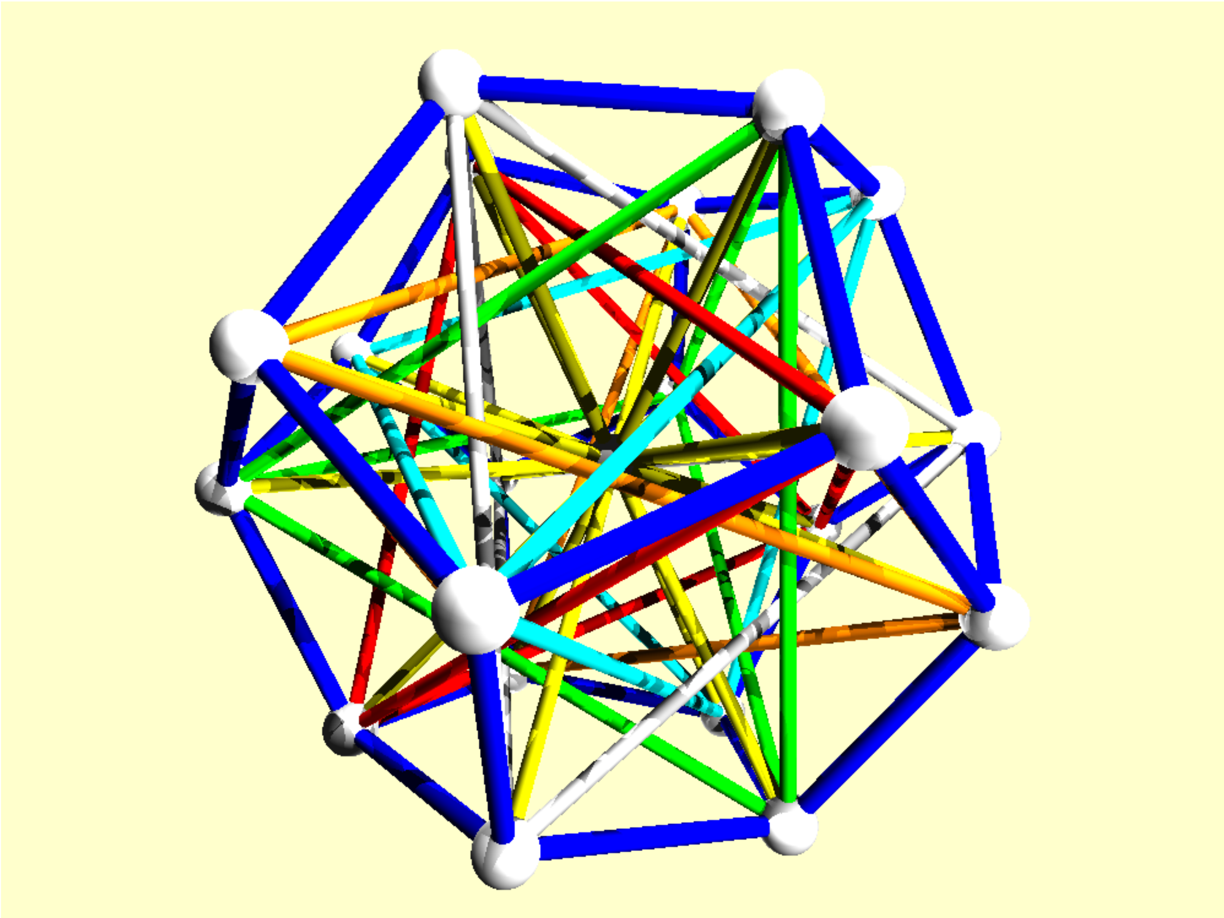

Searching for geometries with MAGMA

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Incidence geometries

- An incidence geometry (X, *, t, I) is a 4-tuple s.t.
  - X is a set of elements;
  - I is a set of types;
  - t : X -> I : x -> t(x) is the type function;
  - * is an incidence relation, i.e. a reflexive, symmetric relation on X x X, such that two distinct elements of the same type cannot be incident;

  The incidence graph is the graph (X,E) where {x,y} ∈ E iff x*y.

  Each maximal clique in the incidence graph contains an element of each type.

- #I is called the rank of the geometry.

Some definitions

- A clique of the incidence graph is called a flag.
- If a flag contains an element of each type, it is called a chamber.
- The number of elements of a flag *F* is the rank of F.
- A geometry is firm (resp. thin) if every flag of corank 1 is contained in at least (resp. exactly) two chambers.

- The residue of a flag F is the geometry whose elements are all the elements incident to F but not in F, with induced incidence relation and type function.

- A geometry is residually connected if every residue of rank at least two has a connected incidence graph.

- A geometry is flag-transitive if its automorphism group acts transitively on the chambers.

Coset geometries

Jacques Tits' algorithm

Take a group G and a set $\{G_i \mid i = 1,\dots, n\}$
of subgroups of G

$X := \{G_i g \mid G_i < G \text{ for all } i \text{ in } \{1..n\}, g \in G\}$

$I := \{1, \dots, n\}$

$t : X \rightarrow I : G_i g \rightarrow i$

* : Two cosets are incident provided their intersection is non-empty.

(X, *, t, I) is a geometry provided that every maximal clique of the incidence graph is of cardinality n.

Coset geometries

Coset geometries from sporadic simple groups

- Ten smallest sporadic groups
- Some results for the O'Nan group
- Some results for Suz
- Suzuki simple groups
- PSL(2,q) groups in rank 2
- …

First step : subgroup lattice computation

G	Order of G	Factored order of G	Degree of G
M_{11}	7920	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	11
M_{12}	95040	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	12
J_1	175560	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	266
M_{22}	443520	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	22
J_2	604800	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	100
M_{23}	10200960	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	23
HS	44352000	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	100
J_3	50232960	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$	6156
M_{24}	244823040	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	24
McL	898128000	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	275
He	4030387200	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	2058
Ru	145926144000	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$	4060
Suz	448345497600	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	1782
O'N	460815505920	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$	122760
Co_3	495766656000	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	276
Co_2	42305421312000	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	2300
Fi_{22}	64561751654400	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	3510
HN	273030912000000	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$	1140000
Ly	51765179004000000	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$	8835156
Th	90745943887872000	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$	143127000
Fi_{23}	4089470473293004800	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$	31671
Co_1	4157776806543360000	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$	98280
J_4	86775571046077562880	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$	173067389
Fi'_{24}	1255205709190661721292800	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$	920808
BM	4154781481226426191177580544000000	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$	13571955000
M	808017424794512875886459904961710757005754368000000000	$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$	97239461142009186000

Table 1: Sporadic groups, their orders and smallest permutation representation degree

G	Factored order of G	Degree of G	Reference
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	11	Buekenhout, LNM 1181, 1984
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	12	Buekenhout-Rees, Math. Comput., 1988
J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	266	Buekenhout, LNM 1181, 1984
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	22	Pfeiffer, Exp. Math., 1997
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	100	Pahlings, 1987
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	23	Pfeiffer, Exp. Math., 1997
HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	100	
J_3	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$	6156	Pfeiffer, 1991
M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	24	Pfeiffer, Exp. Math., 1997
McL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	275	Pfeiffer, Exp. Math., 1997
He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	2058	Merkwitz, 1997
Ru	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$	4060	
Suz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	1782	
O′N	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$	122760	Holt, 1998 (subgroups only)
Co_3	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	276	Merkwitz, 1997
Co_2	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	2300	
Fi_{22}	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	3510	
HN	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$	1140000	
Ly	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$	8835156	
Th	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$	143127000	
Fi_{23}	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$	31671	
Co_1	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$	98280	
J_4	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$	173067389	
Fi'_{24}	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$	920808	
BM	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$	13571955000	
M	$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$	97239461142009186000	

Table 2: Sporadic groups and what is known about their subgroup lattice

Timings on a Pentium Xeon Dual Core at 3.2Ghz (4Mb of cache) with 16Gb of Ram

G	Order(G)	Deg(G)	$cc(G)$	$n(G)$	CPU Time
M_{11}	7,920	11	39	8,651	0.1s
M_{12}	95,040	12	147	214,871	0.41s
J_1	175,560	266	40	158,485	0.15s
M_{22}	443,520	22	156	941,627	0.47s
J_2	604,800	100	146	1,104,344	0.63s
M_{23}	10,200,960	23	204	17,318,406	0.8s
HS	44,352,000	100	589	149,985,646	5.09s
J_3	50,232,960	6156	137	71,564,248	17.46s
M_{24}	244,823,040	24	1529	1,363,957,253	73.94s
McL	898,128,000	275	373	1,719,739,392	4.51s
He	4,030,387,200	2058	1698	22,303,017,686	177.06s
Ru	145,926,144,000	4060	6035	963,226,363,401	**20117.720s**
Suz	448,345,497,600	1782	6381	4,057,939,316,149	**16130.870s**
O'N	460,815,505,920	122760	581	1,169,254,703,685	**7600s**
Co_3	495,766,656,000	276	2483	2,547,911,497,738	67.92s
Fi_{22}	64,561,751,654,400	3510	111004		**7.3 days**

Table 1: Computing times of subgroup lattices

Coset geometries

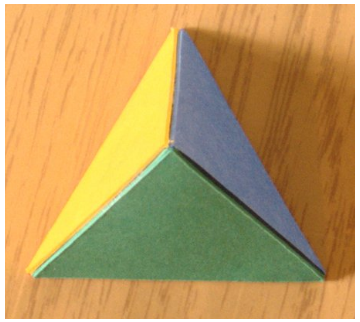
Link with locally 2-arc-transitive graphs

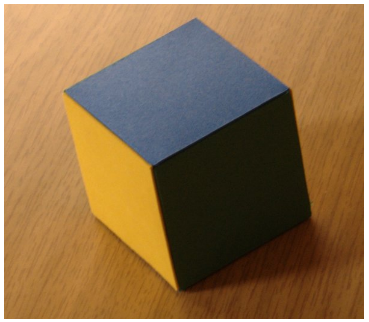
– Our geometries in rank 2 are locally 2-arc-transitive bipartite graphs with the stabilizer of a vertex of one bipart maximal in the group

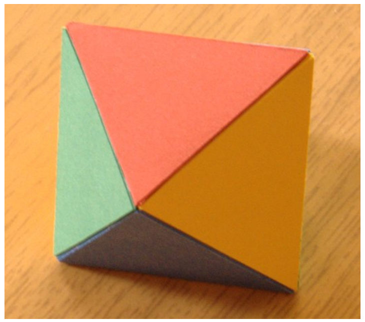
- All locally 2-arc-transitive graphs for the 14 smallest sporadic simple groups are known (Leemans, J. Alg 2009)
- Several locally 2-arc-transitive graphs for PSL(2,q) and Sz(q).

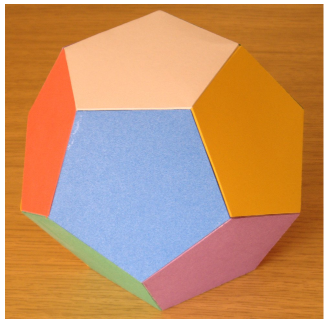
Polytopes in dimension 3
Polyhedra

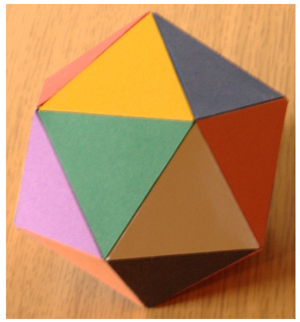
Platonic solids



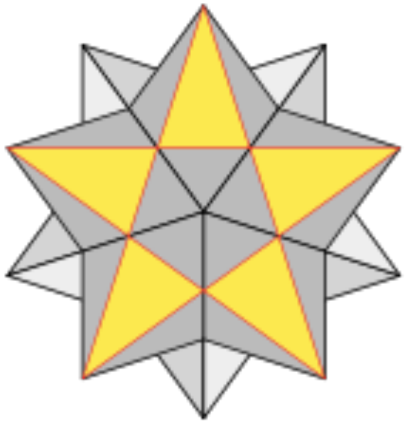








The Kepler-Poinsot Polyhedra



{5/2, 5}

Small stellated dodecahedron

Face: pentagram

{5/2, 3}

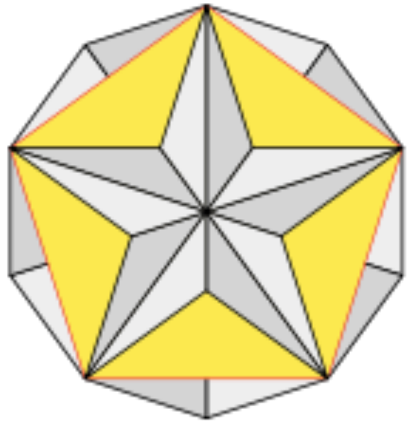
Great stellated dodecahedron

Face: pentagram

{3, 5/2}

Great icosahedron

Face: triangle

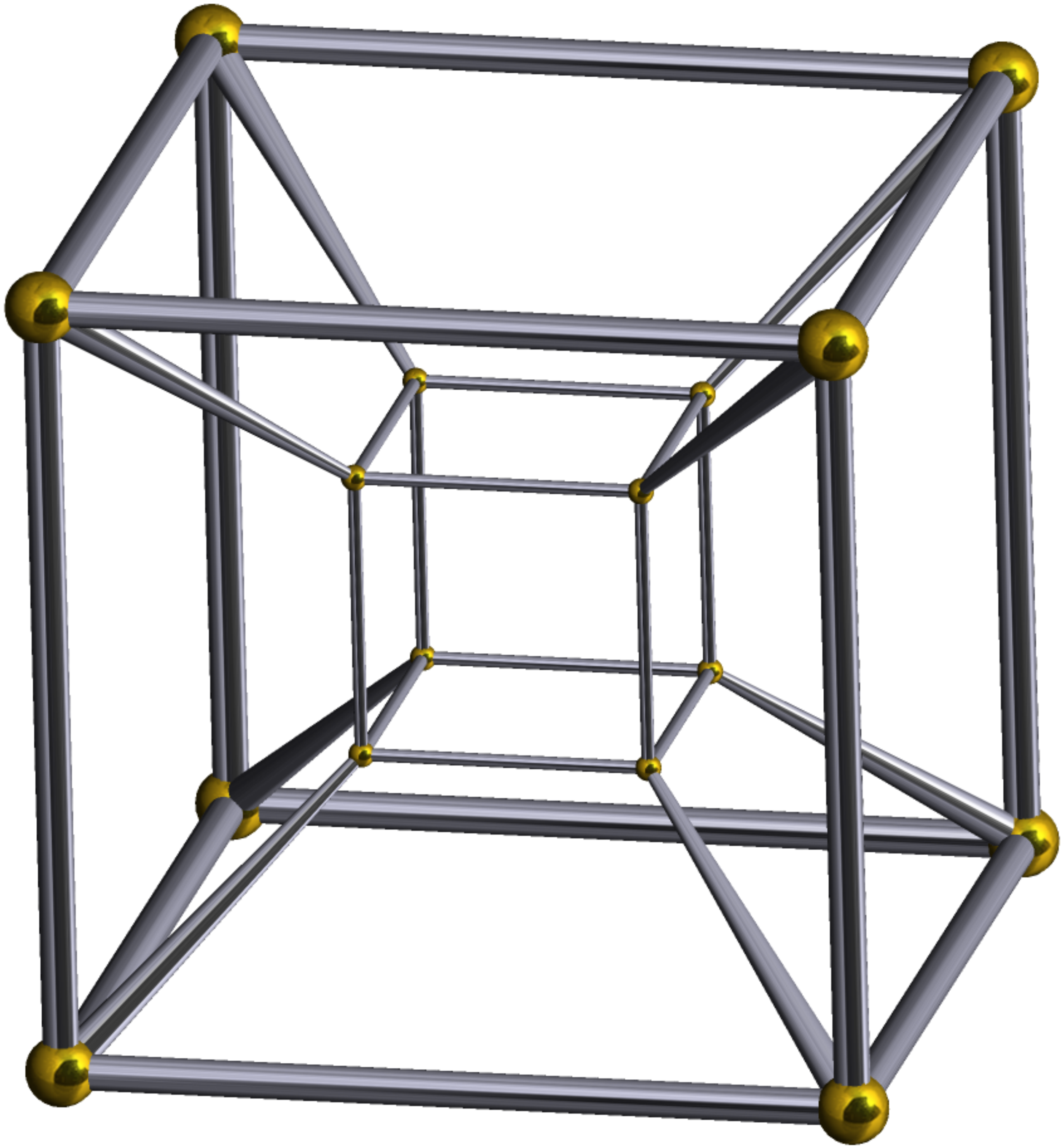


{5, 5/2}

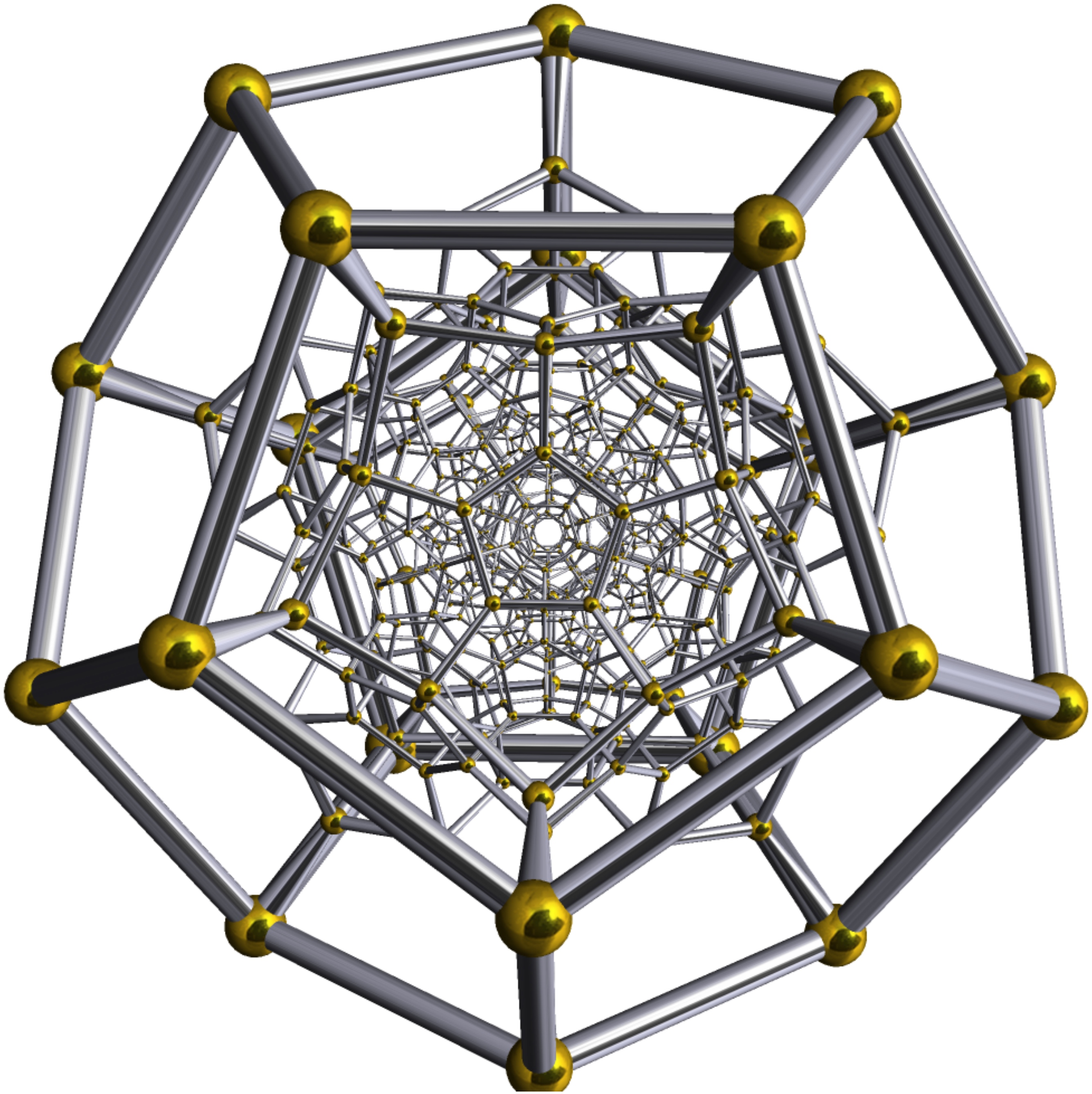
Great dodecahedron

Face: pentagon

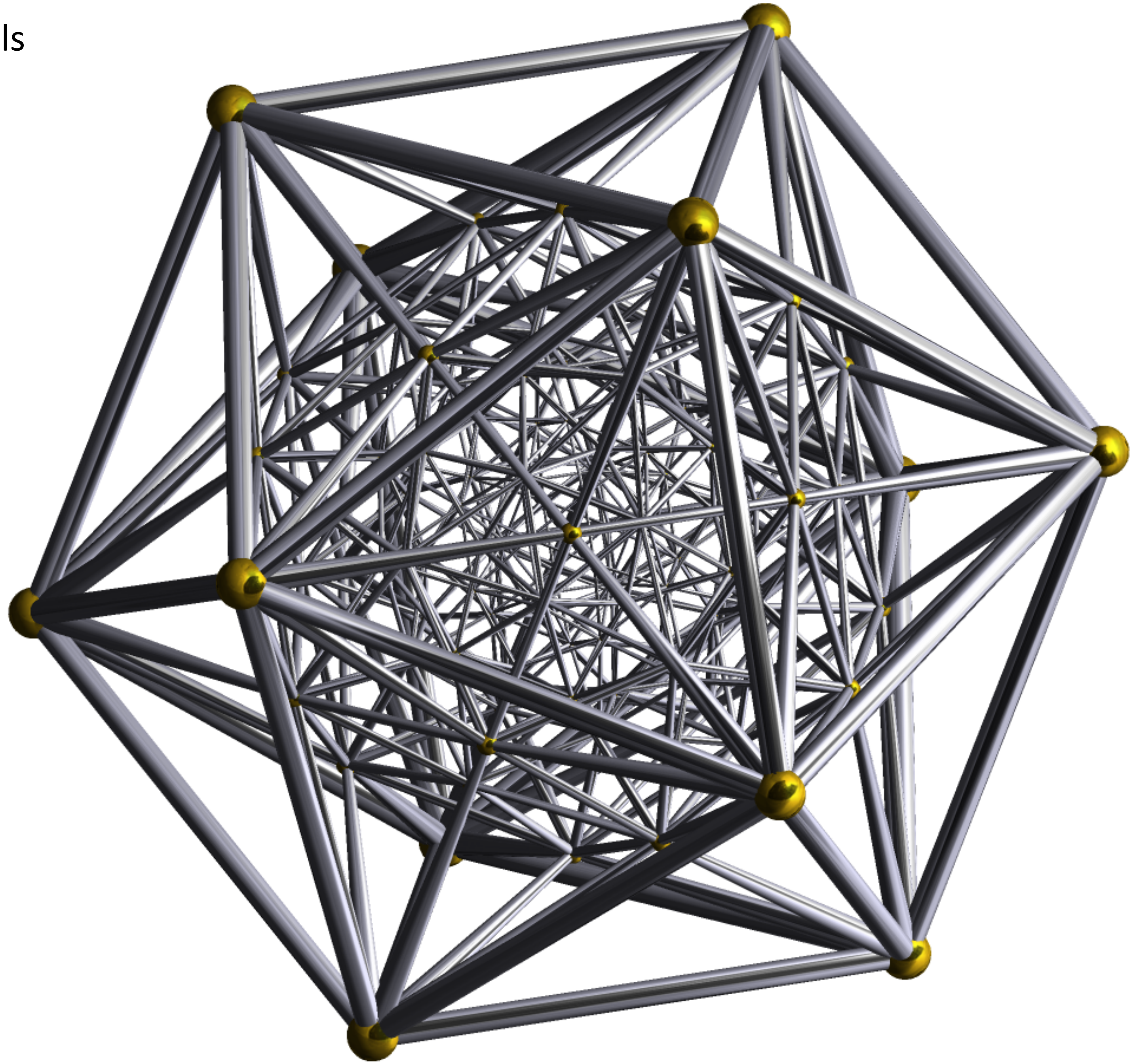
Hypercube



120-cells



600-cells



Abstract polytopes

An abstract *n*-polytope *P* is a partially ordered set satisfying:

A1. It has a least face and a greatest face.

A2. Every flag has exactly $n + 2$ faces.

A3. It is strongly connected.

A4. For each $I = 0, \ldots, n\text{-}1$, if F and G are incident faces of P, of ranks $i\text{-}1$ and $i\text{+}1$ respectively, there are exactly two i-faces H of P such that $F < H < G$.

Abstract regular polytopes

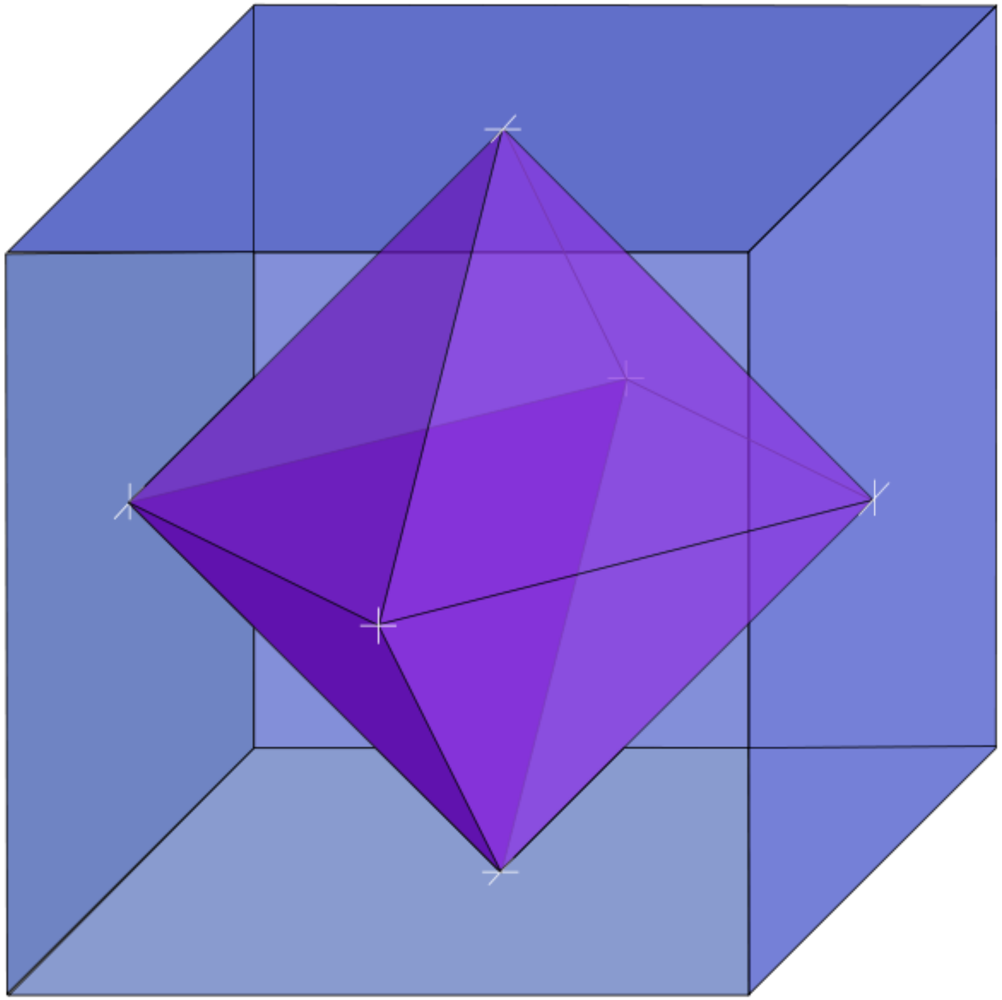
An abstract n-polytope P is regular if its automorphism group G is transitive on the flags of P.

If the automorphism group G has two orbits on the flags such that any two adjacent flags are in distinct orbits, the polytope is called chiral or chirally regular.

Platonic solids and their automorphism group

Sym(4)   Sym(4) x 2   Alt(5) x 2

The automorphism group of the cube (octahedron)



The automorphism group of the cube as an abstract group

Take three involutions *a*, *b*, and *c*

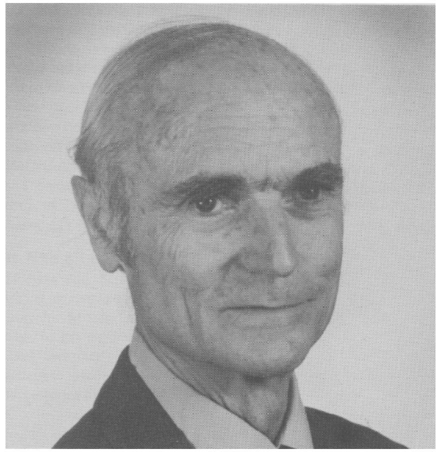
Impose that $(a*b)^3 = (a*c)^2 = (b*c)^4 = 1$

The group <a,b,c> is of finite order.

It is of order 48.

It is isomorphic to the full automorphism group of the cube.

Coxeter groups



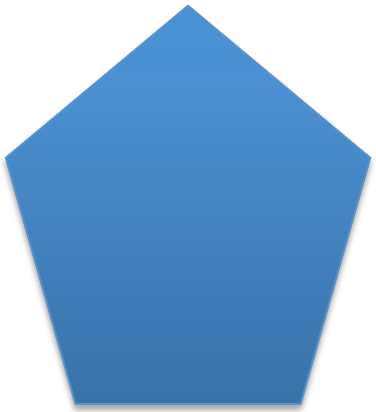
A Coxeter group is a group
$G = < \rho_0, \ldots, \rho_{n\text{-}1} \mid (\rho_i \rho_j)^{m_{ij}} = 1 >$ where
- $m_{ii} = 1$
- $m_{ij} = m_{ji} > 1$ is a positive integer or infinity, whenever $i \neq j$.

The Coxeter matrix associated to G is the matrix (m_{ij}).

Examples of Coxeter groups

$<a,b \mid a^2 = b^2 = (a*b)^n = 1>$ is the dihedral group of order $2n$ for any positive integer n.

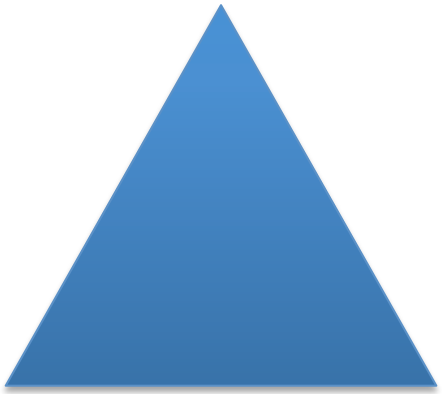
Example for $n = 5$:

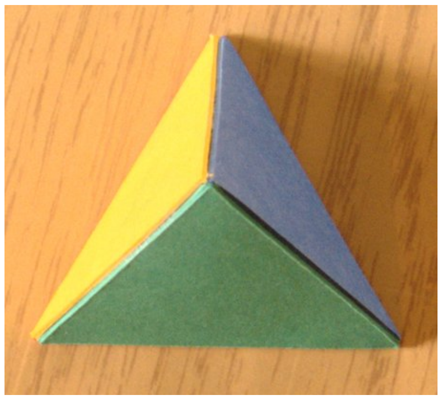


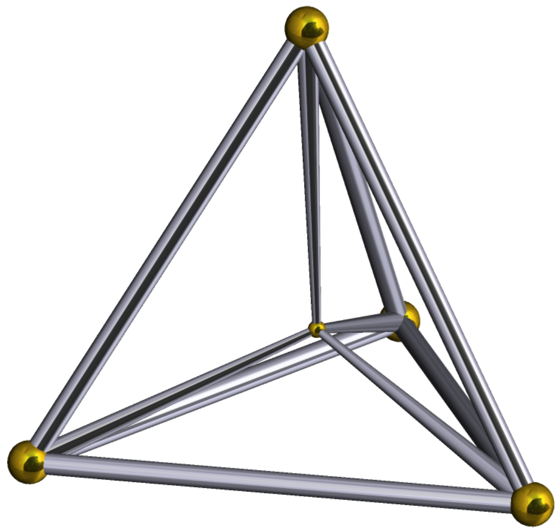
Examples of Coxeter groups

$< \rho_0, \ldots, \rho_{n-1} \mid \rho_i^2 = (\rho_i \rho_{i+2})^2 = (\rho_i \rho_{i+1})^3 = 1 >$
is the symmetric group *Sym(n)*

(E. H. Moore, 1896)







Coxeter diagram

Given a Coxeter group G generated by pairwise distinct involutions $\rho_0, \ldots, \rho_{n-1}$, with associated Coxeter matrix (m_{ij}), the Coxeter diagram is a graph on n vertices. Two vertices i and j are joined by an edge provided that $m_{ij} > 2$. Moreover, on the edge $\{i, j\}$, we write m_{ij} provided $m_{ij} > 3$.

Platonic solids and their Coxeter diagram

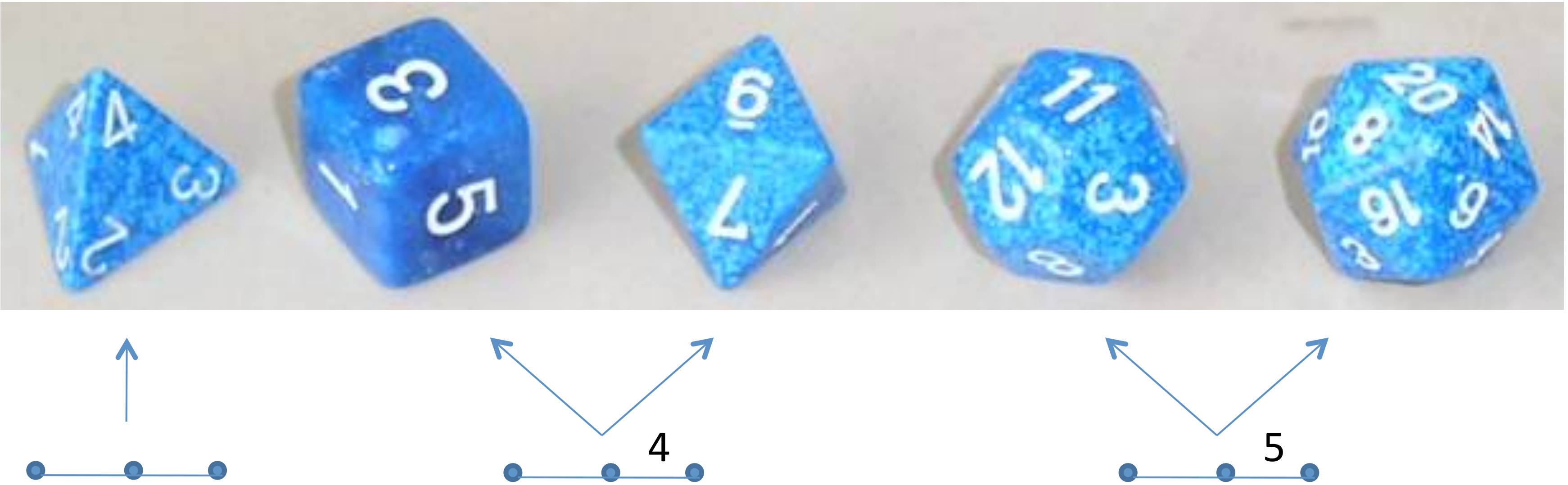
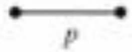
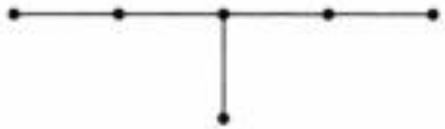
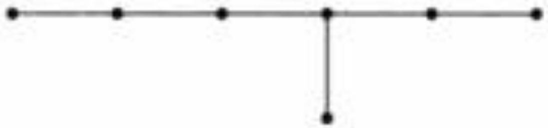


Table 3B1. *The Finite Irreducible Coxeter Groups*

Notation	Diagram	Order
A_n ($n \geq 1$)		$(n+1)!$
B_n ($=D_n$) ($n \geq 4$)		$2^{n-1}n!$
C_n ($=B_n$) ($n \geq 2$)		$2^n n!$
D_2^p ($=I_2(p)$) ($p \geq 3$)		$2p$
E_6		$72 \cdot 6!$
E_7		$8 \cdot 9!$
E_8		$192 \cdot 10!$
F_4		1152
G_3 ($= H_3$)		120
G_4 ($= H_4$)		14400

C-groups

A C-group is a group G generated by pairwise distinct involutions $\rho_0, \ldots, \rho_{n-1}$ which satisfy the following property, called the intersection property:

$\forall J,K \subseteq \{0, \ldots, n-1\}$,

$\langle \rho_j \mid j \in J \rangle \cap \langle \rho_k \mid k \in K \rangle = \langle \rho_j \mid j \in J \cap K \rangle$.

String C-groups

A C-group (G, {ρ_0, . . . , ρ_{n-1}}) is a string C-group if its generators satisfy the following relations.

$$(\rho_j \rho_k)^2 = 1_G$$

$$\forall\ j, k \in \{0, \ldots n-1\} \text{ with } |\, j-k \,| > 1$$

Abstract regular polytopes and String C-groups

String C-groups and abstract regular polytopes are the “same objects”

(one may construct an ARP from a SCG in a unique way and vice-versa)

Same objects with different names

Thin residually connected regular geometries
with a linear diagram

Abstract regular polytopes

String C-groups

What made me work in ARP ?

$G := < s_0, s_1, s_2, s_3 >$

[5,3,5]

G is infinite

[5,3,5] + $(s_0s_1s_2)$^5 = 1

+ $(s_1s_2s_3)$^5 = 1

Coxeter 57-cells and G = PSL(2,19)

[5,3,5] + $(s_0s_1s_2)$^5 = 1

G = J_1 x PSL(2,19)

Atlases of polytopes

- L.-Vauthier : all polytopes with automorphism group *G* such that *S <= G <= Aut(S)*, with *S* a simple group in the Atlas of Finite Groups, of order less than 900,000 (Aeq. Math., 2006)

- Hartley : all polytopes with less than 2000 chambers (some values omitted)

- Hartley and Hulpke : atlas of polytopes for sporadic groups (CDM, 2010)

Polytopes Derived from Sporadic Simple Groups - Auxiliary Information - Windows Internet Explorer

http://www.dr-mikes-maths.com/sporpolys/index.html

Polytopes Derived from... | http://zvonka.fmf.uni-lj.si/...

Dr-Mikes-Maths.Com

Michael Hartley's Maths - and More!

Google Search

Polytopes Derived from Sporadic Simple Groups - Auxiliary Information

This web page contains auxiliary information for the article *Polytopes Derived from Sporadic Simple Groups*, submitted to *Experimental Mathematics* in July 2006.

In that article, the twelve smallest sporadic simple groups were studied, and for each one, the polytopes having that group as an automorphism group were enumerated. This page contains links to extra information that could not be adequately contained in the article.

In particular, the generators of the automorphism groups of every polytope discovered are stored in the file sporpolys.tgz (6.7 Mb, uncompresses to 18 Mb), as permutations. These may be read into GAP via the command

Atlas of Small Regular Polytopes - Windows Internet Explorer

http://www.abstract-polytopes.com/atlas/index.html

The Atlas of Small Regular Polytopes

Questions?
See the FAQ
or other info.

This atlas contains information about all regular polytopes with *n* flags where *n* is at most 2000, and not equal to 1024 or 1536

Feel free to browse!

Polytopes of :

- Rank 1 - **1** nondegenerate and **0** degenerate polytopes
- Rank 2 - **996** nondegenerate and **1** degenerate polytopes
- Rank 3 - **5946** nondegenerate and **993** degenerate polytopes
- Rank 4 - **2912** nondegenerate and **6336** degenerate polytopes
- Rank 5 - **352** nondegenerate and **8599** degenerate polytopes
- Rank 6 - **2** nondegenerate and **6119** degenerate polytopes
- Rank 7 - **0** nondegenerate and **2812** degenerate polytopes
- Rank 8 - **0** nondegenerate and **797** degenerate polytopes
- Rank 9 - **0** nondegenerate and **108** degenerate polytopes

Automorphism Groups of Size :

2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 | 28 | 30 | 32 | 34 | 36 | 38 | 40 | 42 | 44 | 46 | 48 | **50** | 52 | 54 | 56 | 58 | 60 | 62 | 64 | 66 | 68 | 70 | 72 | 74 | 76 | 78 | 80 | 82 | 84 | 86 | 88 | 90 | 92 | 94 | 96 | 98 | **100** | 102 | 104 | 106 | 108 | 110 | 112 | 114 | 116 | 118 | 120 | 122 | 124 | 126 | 128 | 130 | 132 | 134 | 136 | 138 | 140 | 142 | 144 | 146 | 148 | **150** | 152 | 154 | 156 | 158 | 160 | 162 | 164 | 166 | 168 | 170 | 172 | 174 | 176 | 178 | 180 | 182 | 184 | 186 | 188 | 190 | 192 | 194 | 196 | 198 | **200** | 202 | 204 | 206 | 208 | 210 | 212 | 214 | 216 | 218 | 220 | 222 | 224 | 226 | 228 | 230 | 232 | 234 | 236 | 238 | 240 | 242 | 244 | 246 | 248 | **250** | 252 | 254 | 256 | 258 | 260 | 262 | 264 | 266 | 268 | 270 | 272 | 274 | 276 | 278 | 280 | 282 | 284 | 286 | 288 | 290 | 292 | 294 | 296 | 298 | **300** | 302 | 304 | 306 | 308 | 310 | 312 | 314 | 316 | 318 | 320 | 322 | 324 | 326 | 328 | 330 | 332 | 334 | 336 | 338 | 340 | 342 | 344 | 346 | 348 | **350** | 352 | 354 | 356 | 358 | 360 | 362 | 364 | 366 | 368 | 370 | 372 | 374 | 376 | 378 | 380 | 382 | 384 | 386 | 388 | 390 | 392 | 394 | 396 | 398 | **400** | 402 | 404 | 406 | 408 | 410 | 412 | 414 | 416 | 418 | 420 | 422 | 424 | 426 | 428 | 430 | 432 | 434 | 436 | 438 | 440 | 442 | 444 | 446 | 448 | **450** | 452 | 454 | 456 | 458 | 460 | 462 | 464 | 466 | 468 | 470 | 472 | 474 | 476 | 478 | 480 | 482 | 484 | 486 | 488 | 490 | 492 | 494 | 496 | 498 | **500** | 502 | 504 | 506 | 508 | 510 | 512 | 514 | 516 | 518 | 520 | 522 | 524 | 526 | 528 | 530 | 532 | 534 | 536 | 538 | 540 | 542 | 544 | 546 | 548 | **550** | 552 | 554 | 556 | 558 | 560 | 562 | 564 | 566 | 568 | 570 | 572 | 574 | 576 | 578 | 580 | 582 | 584 | 586 | 588 | 590 | 592 | 594 | 596 | 598 | **600** | 602 | 604 | 606 | 608 | 610 | 612 | 614 | 616 | 618 | 620 | 622 | 624 | 626 | 628 | 630 | 632 | 634 | 636 | 638 | 640 | 642 | 644 | 646 | 648 | **650** | 652 | 654 | 656 | 658 | 660 | 662 | 664 | 666 | 668 | 670 | 672 | 674 | 676 | 678 | 680 | 682 | 684 | 686 | 688 | 690 | 692 | 694 | 696 | 698 | **700** | 702 | 704 | 706 | 708 | 710 | 712 | 714 | 716 | 718 | 720 | 722 | 724 | 726 | 728 | 730 | 732 | 734 | 736 | 738 | 740 | 742 | 744 | 746 | 748 | **750** | 752 | 754 | 756 | 758 | 760 | 762 | 764 | 766 | 768 | 770 | 772 |

Terminé Internet | Mode protégé : désactivé 100%

The Leemans-Vauthier atlas

The groups analysed are subdivided into six families:

- Sporadic groups;
- Alternating groups;
- $PSL(2, q)$ groups;
- Some other $PSL(n,q)$ groups;
- Unitary groups;
- Suzuki groups

and their automorphism groups.

Groups of Suzuki type

Theorem (Leemans, Proc. AMS, 2006)
Let $Sz(q) <= G <= Aut(Sz(q))$ with $q = 2^{2e+1}$ and $e > 0$ a positive integer.
Then G is a C-group if and only if $G = Sz(q)$.
Moreover,
if $(G, \{\rho_0, . . . , \rho_{n-1}\})$ is a string C-group, then $n = 3$.

Groups of Suzuki type

If $Sz(q) < G \leq Aut(Sz(q))$, then G is not the automorphism group of an abstract regular polytope;

If $G = Sz(q)$, there exists an abstract regular polytope P such that $G = Aut(P)$. Moreover, if P is an abstract regular polytope such that $G = Aut(P)$, then P must be an abstract polyhedron, i.e. a rank three polytope.

Groups of Suzuki type

Theorem (with Kiefer, JCTA 2010, online)

A given Suzuki group $Sz(q)$ with $q = 2^{2e+1}$ and $e \geq 0$, acts on

$$\frac{1}{2} \sum_{2f+1 | 2e+1} \mu\left(\frac{2e+1}{2f+1}\right) \sum_{\substack{n | 2f+1 \\ n \neq 1}} \lambda(n) \psi(n, 2f+1)$$

polyhedra up to isomorphism and duality, where

$$\lambda(n) = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) \cdot 2^d$$

$$\psi(n, 2f+1) = \sum_{m | \frac{2f+1}{n}} \frac{\sum_{d|m} \mu\left(\frac{m}{d}\right)(2^{nd} - 1)}{m}$$

Observations for PSL(2,q)

- for $q = 7$ and $q = 9$, there is no polytope;

- for all other values of q we tested, there are polytopes of rank three;

- there are polytopes of rank > 3 only for $q = 11$ and 19. In the latter cases, there is exactly one polytope of rank four for each value of q and no polytope of higher rank.

Groups of type PSL(2,q)

It is already known when a *PSL(2, q)* group has rank three polytopes.

Theorem (Sjerve and Cherkassoff, CMR Proc. Lect. Notes, 1993)
The *PSL(2, q)* group may be generated by three involutions, two of which commute, if and only if $q \neq 2, 3, 7$ or 9.

Theorem (L.-Vauthier, Aeq. Math, 2006)
Let $G = PSL(2, q)$. If $(G, \{\rho_0, \ldots, \rho_{n-1}\})$ is a string C-group, then $n < 5$.

Groups and polyhedra

Mazurov, 1980 : which finite non-abelian simple groups are generated by three involutions, two of which commute ? (Kourovka Notebook)

Theorem (Nuzhin, Algebra and Logic, 1997)
Non-sporadic : all but
-Alt(6), Alt(7), Alt(8)
-Lie groups of char 2: $A_2(q)$,$^2A_2(q)$,$A_3(q)$,$^2A_3(q)$
-Lie groups of odd char:$A_2(q)$,$^2A_2(q)$,$A_1(7)$,$A_1(9)$,$B_2(3)$

Sporadic : not known ?

Groups of type PSL(2,q)

Theorem (Leemans - Schulte, Adv. Geom. 2007)
Let $G = PSL(2, q)$.
If $(G, \{\rho_0, \ldots, \rho_3\})$ is a string C-group, then

$$q = 11 \text{ or } 19.$$

Or in other words :

Let $G = PSL(2, q)$. If P is a polytope of rank four on which G acts regularly, then $q = 11$ or 19.

Groups of type PSL(2,q)

- q = 11 gives the 11-cells (Grünbaum, 1977);

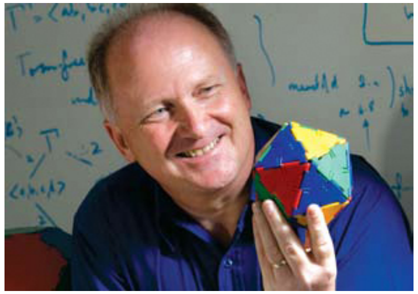
- q = 19 gives the 57-cells (Coxeter, 1982);

- No other thanks to our result.

Groups of type PSL(2,q)

Conder, Potocnik and Siran, 2008

- enumeration of all regular hypermaps for PSL(2,q) and PGL(2,q)



Groups of type PGL(2,q)

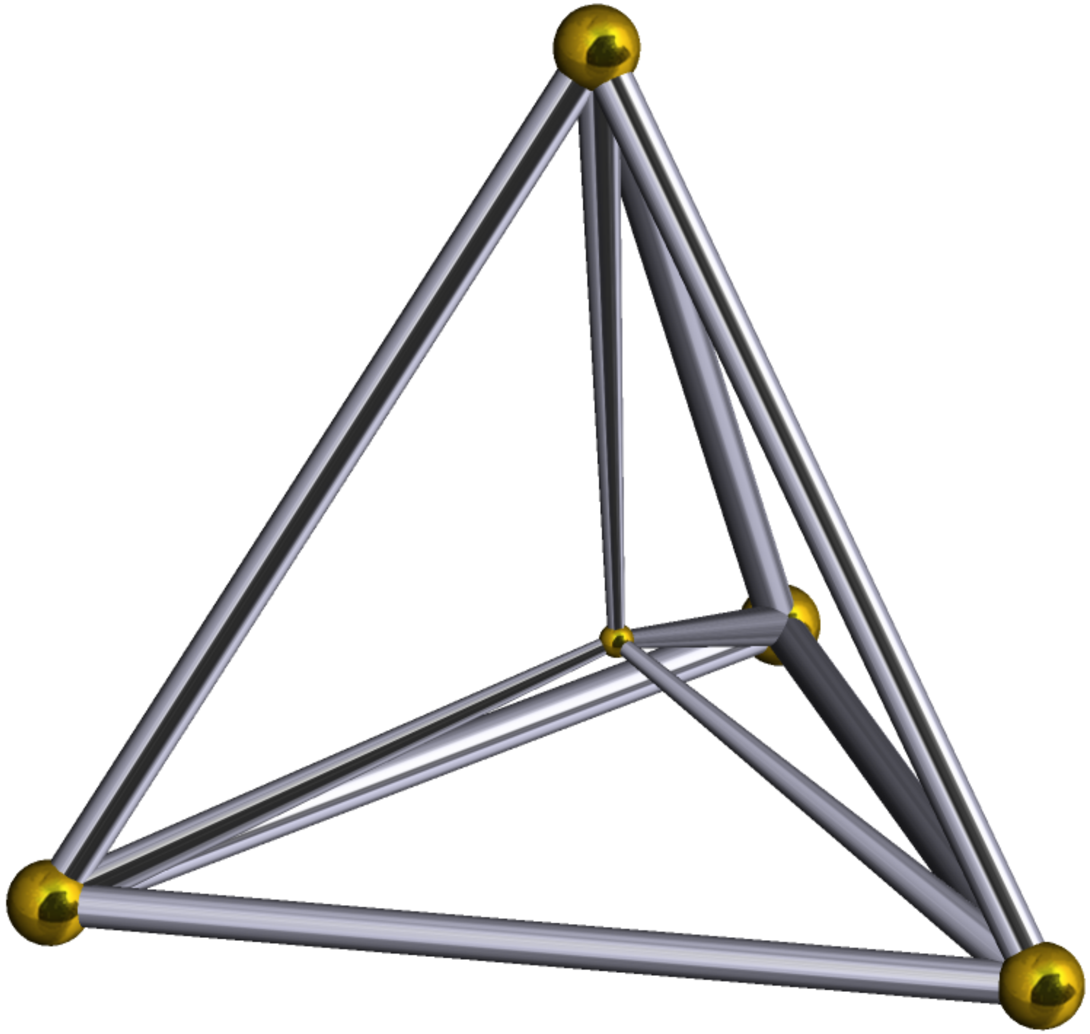
Theorem (Leemans–Schulte, Ars Math Contemp '09)
Let $G = PGL(2, q)$.
If $(G, \{\rho_0, \ldots, \rho_{n-1}\})$ is a string C-group, then $n < 5$.
Moreover, if $n=4$, then $q = 5$.

Or in other words :

Let $G = PGL(2, q)$. If P is a polytope of rank four on which G acts regularly, then $q = 5$.

Observe that Sym(5) and PGL(2,5) are isomorphic.

4-simplex



PSL(2,q) <= G <= PΓL(2,q)

With Buekenhout, De Saedeleer, Hubard and Pellicer (work in progress)

Find similar results as for PSL(2,q) and PGL(2,q)

Groups of type PSL(3,q)

Theorem (Brooksbank, Vincinsky, DCG, 2010)

PSL(3,q) does not act as a regular automorphism group on a polytope.

Groups of Ree type

With Hendrik Van Maldeghem and Egon Schulte.
(work in progress)

The maximum rank is < 5.
We have examples in rank three all the time.
No example in rank four yet.

Symmetric groups

With Maria-Elisa Fernandes (submitted)

1. Sym(n) has regular polytopes of rank at most n-1 and the (n-1)-simplex is the only polytope of rank n-1 for Sym(n);
2. There is a unique regular polytope of rank n-2 for Sym(n) provided $n > 6$;
3. There are regular polytopes of rank 3, ..., n-1 for Sym(n).

Symmetric groups

With Ann Kiefer (preprint)

Theorem 1.1. *Let $n > 3$ be a positive integer. Set $\lambda(k)$, $\psi(k,n)$ and $\eta(n)$ as follows.*

$$\lambda(k) = \begin{cases} \frac{k^2}{4} + k + 1 & \text{if } k \text{ even,} \\ \frac{k^2}{4} + k + \frac{3}{4} & \text{if } k \text{ odd.} \end{cases}$$

$$\psi(k,n) = \begin{cases} \left[\frac{1}{2}\left(k - \lfloor\frac{n-2k}{2}\rfloor\right)\right]^2 + \frac{1}{2}\left(k - \lfloor\frac{n-2k}{2}\rfloor\right) & \text{if } n \equiv 0, 1 \bmod 4, \\ \\ \left[\frac{1}{2}\left(k - \lfloor\frac{n-2k}{2}\rfloor - 1\right)\right]^2 + k - \lfloor\frac{n-2k}{2}\rfloor & \text{if } n \equiv 2, 3 \bmod 4, \end{cases}$$

$$\eta(n) = \begin{cases} \lfloor\frac{n}{4}\rfloor + 1 \text{ if } n \text{ is even,} \\ \\ \lfloor\frac{n-1}{4}\rfloor + 1 \text{ if } n \text{ is odd.} \end{cases}$$

There are, up to isomorphism,

$$-\frac{3}{2} \cdot \lfloor\frac{n}{2}\rfloor + \sum_{k=1}^{\lfloor\frac{n}{2}\rfloor} \lambda(k) \cdot \left(\frac{1}{2}\lfloor\frac{n-2k}{2}\rfloor + 1\right) - \frac{1}{2} \cdot \sum_{k=\eta(n)}^{\lfloor\frac{n}{2}\rfloor} \psi(k,n)$$

pairs of commuting involutions in $Sym(n)$, with $n \neq 6$.

If $n = 6$, there are, up to isomorphism, five pairs of commuting involutions in $Sym(n)$.

Alternating groups

With Maria-Elisa Fernandes and Mark Mixer
(work in progress)

Find the maximum rank of a regular polytope for Alt(n).

Alternating groups

With Ann Kiefer (preprint)

Theorem 1.2. *Let $n > 3$ be a positive integer. Set $\lambda''(k)$ and $\mu(n)$ as follows.*

$$\lambda''(k) = \begin{cases} \lambda(k) - 1 & \text{if } k \leq \nu(n), \\ \lambda(k) - \psi(k,n) - 1 & \text{if } k > \nu(n), \end{cases}$$

$$\mu(n) = -2 \cdot \lfloor \frac{n}{4} \rfloor + \sum_{\substack{k=1 \\ k \text{ even}}}^{\lfloor \frac{n}{2} \rfloor} \left[\lambda_e(k) \cdot \lceil \frac{1}{2} \cdot (\lfloor \frac{n-2k}{2} \rfloor + 1) \rceil + \lambda_o(k) \cdot \lfloor \frac{1}{2} \cdot (\lfloor \frac{n-2k}{2} \rfloor + 1) \rfloor \right]$$

and

$$\lambda_e(k) = \frac{k^2}{8} + \frac{3k}{4} + 1,$$
$$\lambda_o(k) = \frac{k^2}{8} + \frac{k}{4}.$$

Then in $Alt(n)$, with $n \neq 6$, there are, up to isomorphism,

$$\frac{1}{2} \left(\mu(n) + \sum_{\substack{k=1 \\ k \text{ even}}}^{\lfloor \frac{n}{2} \rfloor} \lambda''(k) \right)$$

pairs of commuting involutions.

If $n = 6$, there is, up to isomorphism, a unique pair of commuting involutions in $Alt(n)$.

Alternating groups

Theorem (Fernandez-Leemans-Mixer, 2010)

For each rank $k \geq 3$, there is a regular k-polytope P with automorphism group G(P) isomorphic to an alternating group Alt(n) for some n.

In particular, for each even rank $r \geq 4$, there is a regular polytope with Schläfli type $\{10, 3^{r-2}\}$ and group isomorphic to Alt($2r + 1$), and for each odd rank $q \geq 5$, there is a regular polytope with Schläfli type $\{10, 3^{q-4}, 6, 4\}$ and group isomorphic to Alt($2q + 3$).

Alternating groups

With Maria-Elisa Fernandes and Mark Mixer :

Polytopes of rank r for Alt(2r+1) and Alt(2r+2)

If you want to know … attend the next talk ! ☺

Abstract regular polytopes

An abstract *n*-polytope *P* is regular if its automorphism group *G* is transitive on the flags of *P*.

If the automorphism group *G* has two orbits on the flags such that any two adjacent flags are in distinct orbits, the polytope is called chiral or chirally regular.

Highest rank of a chiral polytope

Pellicer, 2010 (Discrete Math)

constructs a family of chiral polytopes of rank d for every $d > 2$.

Open question

Are there more chiral polytopes than regular polytopes ?

More in terms of what ?

Order of Automorphism group

Genus

Rank

...

Marston Conder's atlas of maps :

3378 orientable regular maps of genus $1 < g < 102$

862 non-orientable regular maps of genus $1 < g < 203$

594 chiral orientably-regular maps of genus $1 < g < 102$

Chiral polytopes

Given a group G, find generators $\sigma_1, \ldots, \sigma_{n-1}$ such that

- $G = <\sigma_1, \ldots, \sigma_{n-1}>$

$\tau_{i,j} := \sigma_i \ldots \sigma_j$ is an involution for all $0 < i < j < n$

$\Gamma_J := <\tau_{i,j} \mid i < j+1; i-1 \text{ and } j \in J>$ for $J \subseteq \{0, \ldots, n-1\}$

- The intersection condition becomes :

  $-\Gamma_I \cap \Gamma_J = \Gamma_{I \cap J}$ for every $I, J \subseteq \{0, \ldots, n-1\}$

- No involutory group automorphism of the form

  $\rho : G \rightarrow G$ such that $\rho(\sigma_1) = \sigma_1^{-1}$, $\rho(\sigma_2) = \sigma_1^2\sigma_2$, and $\rho(\sigma_i) = \sigma_i$ for $(2 < i < n)$.

Chiral polytopes

With Isabel Hubard and Michael I. Hartley :

Atlas of chiral polytopes (submitted, 2011)

With Isabel Hubard :

Chiral polytopes related to Suzuki simple groups.

(work in progress)

Almost simple groups

Type	Number of regular	Number of chiral
Sporadic	2404	6998
PSL(2,q)	3904	2050
PSL(3,q)	1840	3998
PSU(3,q)	1202	4222
Sz(q)	14	412

Here, S <= G <= Aut(S), with S of order < 1 million, in the Atlas of finite groups