

On automorphism groups of Type II codes.

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joint work with Dr. Annika Günther

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Binary Codes

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$$C^\perp := \{v \in \mathbb{F}_2^n \mid v \cdot c = \sum_{i=1}^n v_i c_i = 0 \text{ for all } c \in C\}$$

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Clear: $C \subset C^\perp \Rightarrow wt(c) \in 2\mathbb{Z}$ for all $c \in C$.

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Facts:

- ▶ $C = C^\perp \leq \mathbb{F}_2^n \Rightarrow n = 2 \dim(C)$ is even.
- ▶ $C = C^\perp \leq \mathbb{F}_2^n \Rightarrow \mathbf{1} = (1, \dots, 1) \in C$.
- ▶ $C \leq \mathbb{F}_2^n$ Type II $\Rightarrow n \in 8\mathbb{Z}$.

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Question:

- ▶ Is there a self-dual G -invariant code ? $C = C^\perp \in \mathcal{C}(G)$?
- ▶ Is there a Type II code $C \in \mathcal{C}(G)$?

Group ring codes.

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Theorem 2 (Sloane, Thompson)

$\exists C = C^\perp \in \mathcal{C}_{reg}(G)$ of Type II

\Leftrightarrow

$|G| \in 8\mathbb{Z}$ and the Sylow 2-subgroups of G are not cyclic.

Proof of Theorem 1.

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More precisely, write $G = \dot{\cup} \{h_i, h_i g\}$, then with respect to $h_1, h_1 g, h_2, h_2 g, \dots$ C is the rowspace of

$$\begin{array}{cccc} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & \ddots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \end{array}$$

General permutation representations

In joint work with [Annika Günther](#) we treat arbitrary permutation groups $G \leq S_n$.

Theorem A $\exists C = C^\perp \in \mathcal{C}(G) \Leftrightarrow$ condition (E) is satisfied.

(E) every simple $\mathbb{F}_2 G$ -module S with $S \cong S^* = \text{Hom}(S, \mathbb{F}_2)$ occurs in \mathbb{F}_2^n with even multiplicity.

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Remark. Condition (E) is fulfilled, if $|N_G(H_i)/H_i|$ is even where $H_i := \text{Stab}_G(i)$ for $i \in \{1, \dots, n\}$.

Clear. Theorem A implies Theorem 1.

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Theorem B If $C = C^\perp$ is of Type II, then $P(C) \leq \text{Alt}_n$.

Theorem C $\exists C = C^\perp \in \mathcal{C}(G)$ of Type II \Leftrightarrow

- (a) $n \in 8\mathbb{Z}$,
- (b) condition (E) is satisfied, and
- (c) $G \leq \text{Alt}_n$.

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Proof:

- ▶ Condition (E) for group ring codes is equivalent to even group order.
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Proof of Theorem A

\Rightarrow : $\mathbb{F}_2^n/C^\perp \cong \text{Hom}(C, \mathbb{F}_2)$, so if S is a composition factor of C , then S^* is a composition factor of \mathbb{F}_2^n/C^\perp .

$$\underbrace{\mathbb{F}_2^n \supseteq C^\perp}_{S^*} = \underbrace{C \supseteq \{0\}}_S$$

\Leftarrow : $C \subset C^\perp$ maximal self-orthogonal, then C^\perp/C anisotropic and hence semi-simple,

$$C^\perp/C \cong \perp S_j \text{ with } S_j \cong S_j^* \forall j.$$

$S \perp S$ is hyperbolic since $\mathbb{F}_2 = \{x^2 \mid x \in \mathbb{F}_2\}$.

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Orthogonal groups

Let K be any field, $V = K^{2m}$, $q : V \rightarrow K$ a non-degenerate quadratic form of Witt defect 0. This means that there is

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Clear. $\mathrm{Stab}_{O(V, q)}(U) \leq \ker(D)$

Proof of Theorem B

Let $n \in 8\mathbb{Z}$.

$$V := \mathbf{1}^\perp / \langle \mathbf{1} \rangle = \{x + \langle \mathbf{1} \rangle \mid x \in \mathbb{F}_2^n, \text{wt}(x) \in 2\mathbb{Z}\}$$

$$q : V \rightarrow \mathbb{F}_2, q(x + \langle \mathbf{1} \rangle) := \frac{\text{wt}(x)}{2} + 2\mathbb{Z}.$$

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- ▶ The maximal isotropic subspaces of (V, q) are precisely the images of the Type II codes in \mathbb{F}_2^n .
- ▶ S_n fixes $\mathbf{1}$ and preserves the weight hence embeds into $O(V, q)$.
- ▶ The restriction of the Dickson homomorphism $D : S_n \rightarrow \{1, -1\}$ is the sign.

Generalization of Theorem B

This shows more general:

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For odd characteristic, the weight preserving mappings that preserve orthogonality are all permutations and sign changes

$$\{\pm 1\}^n : S_n$$

and one obtains

Theorem B''. Let $p > 2$ and $C = C^\perp \leq \mathbb{F}_{p^d}^n$. Then $\det(\text{Aut}(C)) = \{1\}$.

Proof of Theorem C

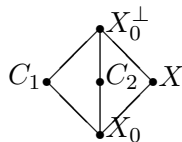
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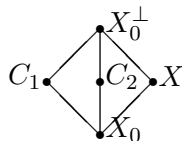
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Then $X_0^\perp/X_0 \cong \mathbb{F}_2 \oplus \mathbb{F}_2$.

- ▶ C_1 and C_2 are doubly-even.
- ▶ $\dim(C_1) - \dim(C_1 \cap C_2) = 1$ is odd.
- ▶ $G \leq P(X) \leq P(X_0)$ acts on $\{C_1, C_2\}$.
- ▶ $D(G) = \{1\}$ so $C_i g = C_i$ for all $g \in G$, $i = 1, 2$.
- ▶ $C_i \in \mathcal{C}(G)$ are Type II.