# Computations around the Automorphism Group of a Free Group

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**(From joint work with Alex Lubotzky (Jerusalem))**

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Example n=2:

$$\mathrm{Aut}^+(F_2) = \langle \alpha, \beta \mid \alpha^4 = \beta^3 = \alpha^2\beta^2\alpha^2\beta\alpha\beta\alpha^2\beta^2\alpha = 1 \rangle \leq_2 \mathrm{Aut}(F_2)$$

$$\alpha(x_1) = x_2^{-1},\ \alpha(x_2) = x_1; \qquad \beta(x_1) = x_1^{-1}x_2^{-1},\ \beta(x_2) = x_1$$

**Linear representations of $\mathrm{Aut}(F_n)$?**

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Theorem ( Formanek, Procesi)

*For $n \geq 3$ the automorphism group $\mathrm{Aut}(F_n)$ is not linear.*

$\mathrm{Aut}(F_2)$ is linear!

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is surjective:

$$\sigma(x_1) = x_2, \quad \sigma(x_2) = x_1, \quad \sigma(x_i) = x_i \ (i = 3, \ldots, n)$$

$$\psi_{ij}(x_i) = x_i x_j, \qquad \psi_{ij}(x_k) = x_k \quad (i \neq j, \ k \neq i)$$

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The images of the $\psi_{ij}$ and $\sigma$ generate $\mathrm{GL}(n, \mathbb{Z})$.

## An easy linear representation of $\mathrm{Aut}(F_n)$ (continued)

$$\langle 1 \rangle \to IA_n \to \mathrm{Aut}(F_n) \to \mathrm{GL}(n, \mathbb{Z}) \to \langle 1 \rangle$$

$IA_n$ is called Torelli group.
$IA_2$=group of inner automorphisms.

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**What about $IA_n$ in general?**

**Theorem (Wilhelm Magnus)**

*$IA_n$ is finitely generated as a group.*

- quotients of the lower central series are torsionfree.
- quotients of the lower central series are known as $\mathrm{GL}(n,\mathbb{Z})$-modules.

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$R$ is free on

$$x_1, \ldots, x_{n-1}, \qquad y x_1 y^{-1}, \ldots, y x_{n-1} y^{-1}, \qquad y^2$$

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$V := \mathbb{Q} \otimes R^{\mathrm{ab}} = \mathbb{Q}^t$ with $t = 2n - 1 = n + (n-1)$.

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$C_2$ acts on $V$ with Eigenpace-Decomposition:

$$V = V^+ \oplus V^-, \qquad V^+ = \mathbb{Q}^n, \quad V^- = \mathbb{Q}^{n-1}$$

## Another easy virtual linear representation of $\mathrm{Aut}(F_n)$ (continued)

$$\Gamma(C_2, \pi) := \{\varphi \in \mathrm{Aut}(F_n) \mid \varphi(R) = R,\ \varphi \text{ induces Id on } F_n/R = G\}$$

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$$\Gamma(C_2,\pi) \to \mathrm{GL}(n,\mathbb{Z}) \times \mathrm{GL}(n-1,\mathbb{Z})$$

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- $[\mathrm{Aut}(F_n) : \Gamma(C_2,\pi)] = 2^n - 1$.
- $\rho_{-1}$ is surjective.
- the kernel of $\rho_{-1}$ is finitely generated (M. Siegmund).

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Yes for $n = 2$ to Problem B.

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Yes for $n = 3$ to Problem B.

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## The Torelli group

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**Theorem**

*Let $n \geq 3$. There is a subgroup $\Gamma \leq \mathrm{IA}(F_n)$ of finite index and a representation*

$$\rho : \Gamma \to \mathrm{GL}((n-1), \mathbb{Z})$$

*such that $\rho(\Gamma)$ has finite index.*

Same Theorem for $\mathrm{Out}(F_n) := \mathrm{Aut}(F_n)/\mathrm{Inner\ Automorphisms}$

**Corollary**

*The groups $\mathrm{IA}(F_3)$, $\mathrm{Out}(F_3)$ are large, i.e. have a subgroup of finite index with a nonabelian free quotient.*

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$$\rho_{G,\pi} : \Gamma(G, \pi) \to \mathrm{GL}(t, \mathbb{Z})$$

# Restrictions

$$\langle 1\rangle \to R \to F_n \to G \to \langle 1\rangle$$

$G$ acts on $\mathbb{C} \otimes_{\mathbb{Z}} \bar{R}$ by conjugation

$$\mathcal{G}_{G,\pi} := \mathrm{Aut}_G(\mathbb{C} \otimes_{\mathbb{Z}} \bar{R}) \leq \mathrm{GL}(t, \mathbb{C})$$

$$\mathcal{G}_{G,\pi}(\mathbb{Z}) := \{\, \phi \in \mathcal{G}_{G,\pi} \mid \phi(\bar{R}) = \bar{R} \,\}$$

$$\rho_{G,\pi}(\Gamma(G,\pi)) \leq \mathcal{G}_{G,\pi}(\mathbb{Z})$$

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$$\mathcal{G}^1_{G,\pi} \leq \mathcal{G}_{G,\pi}$$

the kernel of all $\mathbb{Q}$-defined homomorphisms from the complex algebraic group $\mathbb{C} \otimes \mathcal{G}_{G,\pi}$ to the multiplicative group.

# What is $\mathcal{G}_{G,\pi}$?

**Theorem (Gaschütz)**

$$\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R} \cong \mathbb{Q} \oplus \mathbb{Q}[G]^{n-1}$$

*as G-module.*

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$$\mathcal{G}_{G,\pi}(\mathbb{Q}) = \mathrm{GL}(n, \mathbb{Q}) \times \prod_{i=2}^{\ell} \mathrm{GL}((n-1)h_i, D_i^{\mathrm{op}}).$$

$$\mathcal{G}^1_{G,\pi}(\mathbb{Q}) = \mathrm{SL}(n, \mathbb{Q}) \times \prod_{i=2}^{\ell} \mathrm{SL}((n-1)h_i, D_i^{\mathrm{op}}).$$

## Arithmetic Subgroups

$\mathcal{R}_i \leq D_i^{\mathrm{op}}$ an order

$$\mathcal{G}_{G,\pi}(\mathbb{Z}) \sim \mathrm{GL}(n,\mathbb{Z}) \times \prod_{i=2}^{\ell} \mathrm{GL}((n-1)h_i, \mathcal{R}_i).$$

$$\mathcal{G}^1_{G,\pi}(\mathbb{Z}) \sim \mathrm{SL}(n,\mathbb{Z}) \times \prod_{i=2}^{\ell} \mathrm{SL}((n-1)h_i, \mathcal{R}_i).$$

$$\rho_{G,\pi} : \Gamma(G,\pi) \to \mathrm{GL}(n,\mathbb{Z}) \times \prod_{i=2}^{\ell} \mathrm{GL}((n-1)h_i, \mathcal{R}_i).$$

## Theorem (General)

*Assume $n$ is a natural number with $n \geq 3$. Let $\pi : F_n \to G$ be a redundant presentation of the finite group $G$. Then*

$$\rho_{G,\pi}(\Gamma(G,\pi)) \cap \mathcal{G}^1_{G,\pi}$$

*is of finite index in the arithmetic group $\mathcal{G}^1_{G,\pi}(\mathbb{Z})$.*

## Definition

A presentation $\pi : F_n \to G$ is called redundant if at least one of the generators of $F_n$ is mapped to 1.

### Theorem

*Let $n \geq 2$, $h \geq 1$ be natural numbers. There is a subgroup $\Gamma \leq \mathrm{Aut}(F_n)$ of finite index and a representation*

$$\rho : \Gamma \to \mathrm{SL}((n-1)h, \mathbb{Z})$$

*such that $\rho(\Gamma)$ is of finite index in $\mathrm{SL}((n-1)h, \mathbb{Z})$.*

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$$\rho : \Gamma \to \prod_{i=1}^{k} \mathrm{SL}((n-1)i, \mathbb{Z})$$

*such that $\rho(\Gamma)$ is of finite index in $\prod_{i=1}^{k} \mathrm{SL}((n-1)i, \mathbb{Z})$.*

## Theorem

*Let* $n \geq 2$, $k \geq 1$, $h_1 < \ldots < h_k$, $m_1, \ldots, m_k$ *be natural numbers. Let* $\mathbb{Q}(\zeta_{m_i})$ *be the field of* $m_i$*-th roots of unity and* $\mathbb{Z}(\zeta_{m_i})$ *its ring of integers. There is a subgroup* $\Gamma \leq \mathrm{Aut}(F_n)$ *of finite index and a representation*

$$\rho : \Gamma \to \prod_{i=1}^{k} \mathrm{SL}((n-1)h_i, \mathbb{Q}(\zeta_{m_i}))^{m_i}$$

*such that* $\rho(\Gamma)$ *is commensurable with* $\prod_{i=1}^{k} \mathrm{SL}((n-1)h_i, \mathbb{Z}(\zeta_{m_i}))^{m_i}$.

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**Problem:** (Congruence Subgroup Problem)
Does every subgroup of finite index in $\mathrm{Aut}(F_n)$ contain a $\Gamma(G, \pi)$?

- yes for $n = 2$ (Asada, Rapinchuk, Bux)
- Conjecture: no for $n = 3$.
- Conjecture: yes for $n \geq 4$.

## Subgroups in $\mathrm{Aut}(F_2)$, $\Delta \leq_f \Gamma(G,\pi)$

| $G$ | $\pi(x)$ | $\pi(y)$ | $[\mathrm{Aut}^+(F_2):\Delta]$ | $\Delta^{\mathrm{ab}}$ |
|---|---|---|---|---|
| $C_2$ | (1,2) | (1) | 3 | $\mathbb{Z}^2 \times C_2 \times C_4$ |
| $C_3$ | (1,2,3) | (1) | 8 | $\mathbb{Z} \times C_3^2$ |
| $C_4$ | (1,2,3,4) | (1) | 12 | $\mathbb{Z}^2 \times C_4$ |
| $C_2 \times C_2$ | (1,2) | (3,4) | 6 | $\mathbb{Z}^2 \times C_2^3$ |
| $C_5$ | (1,2,3,4,5) | (1) | 24 | $\mathbb{Z}^3 \times C_5$ |
| $C_6$ | (1,2,3,4,5,6) | (1) | 24 | $\mathbb{Z}^3 \times C_6$ |
| $S_3$ | (1,2,3) | (1,2) | 18 | $\mathbb{Z}^2 \times C_2$ |
| $C_7$ | (1,2,3,4,5,6,7) | (1) | 48 | $\mathbb{Z}^5 \times C_7$ |
| $D_4$ | (1,2,3,4) | (1,4)(2,3) | 24 | $\mathbb{Z}^3 \times C_2$ |
| $D_5$ | (1,2,3,4,5) | (1,5)(2,4) | 30 | $\mathbb{Z}^2 \times C_2$ |
| $A_4$ | (1,2,3) | (1,2)(3,4) | 96 | $\mathbb{Z}^3$ |
| $S_3 \times C_2$ | (1,3)(4,5) | (1,2) | 36 | $\mathbb{Z}^3 \times C_2$ |
| $\mathrm{Sm}(12,1)$ | $\sigma_1$ | $\sigma_2$ | 72 | $\mathbb{Z}^3 \times C_2$ |
| $A_5$ | (1,2,3,4,5) | (1,2,3) | 1080 | $\mathbb{Z}^{17}$ |

## Computations: 2

*Subgroups in* $\mathrm{Aut}(F_2)$

| $G$ | T-System | Length | $[\mathrm{Aut}(F_2) : \Delta]$ | $\Delta^{\mathrm{ab}}$ |
|---|---|---|---|---|
| $\mathrm{PSL}(2,7)$ | p1 | 7 | 2328 | $C_{12} \times \mathbb{Z}$ |
| $\mathrm{PSL}(2,7)$ | p2 | 16 | 10752 | $\mathbb{Z}^{11}$ |
| $\mathrm{PSL}(2,7)$ | p3 | 16 | 10752 | $\mathbb{Z}^{11}$ |
| $\mathrm{PSL}(2,7)$ | p4 | 16 | 12096 | $\mathbb{Z}^{26}$ |
| $\mathbf{A}_5$ | a51 | 10 | 1200 | $\mathbb{Z}^{7}$ |
| $\mathbf{A}_5$ | a52 | 9 | 2160 | $\mathbb{Z}^{17}$ |
| $\mathbf{A}_6$ | a61 | 10 | 7200 | $\mathbb{Z}^{18}$ |
| $\mathbf{A}_6$ | a62 | 12 | 8640 | $\mathbb{Z}^{42}$ |
| $\mathbf{A}_6$ | a63 | 15 | 5400 | $C_2 \times \mathbb{Z}^{13}$ |
| $\mathbf{A}_6$ | a64 | 16 | 11520 | $\mathbb{Z}^{28}$ |

## Problems for $n = 2$

Given $G$ and $\pi : F_2 \to G$, compute

- $[\mathrm{Aut}(F_2) : \Gamma(G, \pi)]$,
- $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes \Gamma(G, \pi)^{\mathrm{ab}})$.

Assume $R = \mathrm{Ker}(\pi)$ is characteristic in $F_2$.

- Describe $\mathbb{C} \otimes \Gamma(G, \pi)^{\mathrm{ab}}$ as a $\mathrm{Aut}(F_2)/\Gamma(G, \pi)$-module.

## Computations: 3

*Subgroups in* $\mathrm{Aut}(F_3)$

| $G$ | $\pi(x)$ | $\pi(y)$ | $[\mathrm{Aut}(F_3) : \Delta]$ | $\Delta^{\mathrm{ab}}$ |
|---|---|---|---|---|
| $C_2$ | (1,2) | (1) | 7 | $C_2^3$ |
| $C_3$ | (1,2,3) | (1) | 26 | $C_6$ |
| $S_3$ | (1,2,3) | (1,2) | 168 | $C_2^3$ |
| $D_4$ | (1,3) | (1,4,3,2) | 336 | $C_2^8$ |
| $Q_8$ | – | – | 336 | $C_2^7$ |
| $D_5$ | (1,2,3,4,5) | (1,5)(2,4) | 840 | $C_2^5$ |
| $A_4$ | (1,2,3) | (1,2)( 3,4) | 1560 | $C_6$ |
| $S_3 \times C_2$ | (1,3)(4,5) | (1,2) | 1008 | $C_2^9$ |
| $\mathrm{Sm}(12,1)$ | $\sigma_1$ | $\sigma_2$ | 1344 | $C_2^2 \times C_4$ |
| $A_5$ | (1,2,3,4,5) | (1,2,3) | 200160 | ? |

## Problems for $n = 3$

Given $G$ and $\pi : F_2 \to G$, compute

- $[\mathrm{Aut}(F_2) : \Gamma(G, \pi)]$,
- $\Gamma(G, \pi)^{\mathrm{ab}})$ and show that it is always finite.

**Theorem**

*$\Gamma(G, \pi)^{\mathrm{ab}}$ is finite if $G$ is solvable.*

- Find a subgroup $\Delta$ of minimal index in $\mathrm{Aut}(F_3)$ with $\Delta^{\mathrm{ab}}$ infinite.

## Computations: 4

*Subgroups in* $\mathrm{Aut}(F_4)$

| $G$ | $\|G\|$ | $[\mathrm{Aut}(F_4) : \Delta]$ | $\Delta^{\mathrm{ab}}$ |
|---|---|---|---|
| $C_2$ | 2 | 15 | $C_2^2$ |
| $C_3$ | 3 | 80 | $C_6$ |
| $S_3$ | 6 | 80 | $C_6$ |
| $D_4$ | 8 | 3360 | $C_2^4$ |
| $Q_8$ | 8 | 840 | $C_2^6$ |
| $D_5$ | 10 | 930 | $C_2^3$ |
| $A_4$ | 12 | 1680 | $C_3 \times C_6$ |
| $S_3 \times C_2$ | 12 | 2730 | $C_2^4$ |
| $\mathrm{Sm}(12, 1)$ | 12 | 3120 | $C_2^2 \times C_4$ |
| $A_5$ | 60 | 213098 | ? |
| $C_2^2$ | 4 | 210 | $C_2^4$ |
| $C_2^3$ | 8 | 2520 | $C_2^{13}$ |
| $C_2^4$ | 16 | 20160 | $C_2^{40}$ |

## Problems for $n = 4$

Given $G$ and $\pi : F_2 \to G$, compute

- $[\mathrm{Aut}(F_2) : \Gamma(G, \pi)]$,
- $\Gamma(G, \pi)^{\mathrm{ab}})$ and show that it is always finite.

**Theorem**

$\Gamma(G, \pi)^{\mathrm{ab}}$ *is finite if* $G$ *is solvable.*

- Is there a subgroup $\Delta$ in $\mathrm{Aut}(F_3)$ with $\Delta^{\mathrm{ab}}$ infinite?

## Non-redundant presentations

$G = \mathbf{A}_5,\ n = 2,\ \pi(x) = (1,2,3,4,5),\ \pi(y) = (1,2,3).$

$$[\mathrm{Aut}(F_2) : \Gamma(G,\pi)] = 1080, \qquad \Gamma(G,\pi)^{\mathrm{ab}} = \mathbb{Z}^{17}, \qquad \Gamma(G,\pi) \to F_9$$

$$\mathbb{Q}[\mathbf{A}_5] = \mathbb{Q} \oplus M_4(\mathbb{Q}) \oplus M_6(\mathbb{Q}) \oplus M_3(\mathbb{Q}[\sqrt{5}])$$

$$\rho : \Gamma(G,\pi) \to \mathrm{GL}(3, \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right])$$

- Zariski-closure of $\rho(\Gamma(G,\pi))$ is inside but not equal to $\mathrm{SL}(3,\mathbb{Q}[\sqrt{5}])$.
- $\rho(\Gamma(G,\pi))$ has a free nonabelian quotient of rank 2,
- $\rho(\Gamma(G,\pi))$ is not an arithmetic subgroup of an algebraic subgroup of $\mathrm{GL}(3,\mathbb{Q}[\sqrt{5}])$

## Problem (SL)

- Is $\rho(\Gamma(G,\pi)) \leq \mathcal{G}^1_{G,\pi}$ (up to finite index)?

If not then Problem A is solved positively.

**Problem (SL)**

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**Theorem**

*If $G$ is metabelian then $\rho(\Gamma(G,\pi)) \leq \mathcal{G}^1_{G,\pi}$ up to finite index.*

# The Mappingclass Group

$$T_{2g} := \langle\, a_1, \ldots, a_g,\ b_1, \ldots, b_g \mid [a_1, b_1][a_2, b_2] \ldots [a_g, b_g] = 1 \,\rangle$$

$$M(2g) := \mathrm{Out}(T_{2g}), \qquad \rho_{2g} : M(2g) \to \mathrm{Sp}(2g, \mathbb{Z})$$

**Theorem**

*There are subgroups of finite index $\Gamma \leq M(2g)$ which can be mapped onto arithmetic subgroups in*

- $\mathrm{Sp}(2(g-1))$,
- *Unitary groups over totally real number fields,*
- $\mathrm{SL}$ *over certain skew fields over* $\mathbb{Q}$.

Same for the Torelli.

**Corollary**

*$M(4)$ and its Torelli subgroup are large.*

## Cyclic Groups

$C_m = \langle g \rangle =$ cyclic group of order $m$, $n \geq 2$

$$\pi : F_n \to C_m, \qquad \pi(x_1) = g,\ \pi(x_2) = 1, \ldots, \pi(x_n) = 1$$

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$$\mathbb{Q}[C_m] = \oplus_{d|m} \mathbb{Q}(\zeta_d)$$

**Theorem**

*The image of the representation* $\rho_{C_m,\pi}$ *is*

$$\mathrm{GL}(n, \mathbb{Z}) \times \prod_{d|m,\, d>1} \mathrm{GL}^+(n-1, \mathbb{Z}(\zeta_d))$$

*where* $\mathrm{GL}^+(n-1, \mathbb{Z}(\zeta_d))$ *is the subgroup consisting of those elements in* $\mathrm{GL}(n-1, \mathbb{Z}(\zeta_d))$ *which have a power of* $\zeta_d$ *as determinant.*

## Proof Theorems from the Beginning

The Choice of $G$ and $\pi : F_n \to G$

- $G$ has to be generated by few elements,
- the rational representation theory of $G$ has to be understood.

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- the rational representation theory of $G$ has to be understood.

$$G = \mathrm{Symmetric}(h) \times C_m \times C_m.$$

## Consequences 1

### Theorem

*There is a subgroup* $\Gamma \leq \mathrm{Aut}(F_3)$ *of finite index and a representation*

$$\rho : \Gamma \to \mathrm{SL}(2, \mathbb{Z})$$

*such that* $\rho(\Gamma)$ *is of finite index in* $\mathrm{SL}(2, \mathbb{Z})$.

### Corollary

*The automorphism group* $\mathrm{Aut}(F_3)$ *is large, that is it has a subgroup of finite index which can be mapped onto a free nonabelian group. In particular* $\mathrm{Aut}(F_3)$ *does not have Kazdhan's property (T).*

## Consequences 2

$\mathcal{A}(\mathrm{Aut}(F_n))$ the pro-algebraic completion of $\mathrm{Aut}(F_n)$.

**Theorem**

*Let $n \geq 2$ be a natural number and $\mathcal{S}(\mathrm{Aut}(F_n))$ be the semisimple part of the connected component of the identity of the pro-algebraic completion $\mathcal{A}(\mathrm{Aut}(F_n))$. Then for every $h \in \mathbb{N}$, the group $\mathrm{SL}((n-1)h, \mathbb{C})$ appears infinitely many times as a factor in $\mathcal{S}(\mathrm{Aut}(F_n))$. That is, there is a surjective homomorphism*

$$\mathcal{S}(\mathrm{Aut}(F_n)) \to \prod_{h=1}^{\infty} \prod_{i=1}^{\infty} \mathrm{SL}((n-1)h, \mathbb{C}).$$

## The Torelli group

$$\langle 1\rangle \to \mathrm{IA}_n \to \mathrm{Aut}(F_n) \to \mathrm{GL}(n,\mathbb{Z}) \to \langle 1\rangle$$

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$$\langle 1\rangle \to \mathrm{IA}_n \to \mathrm{Aut}(F_n) \to \mathrm{GL}(n,\mathbb{Z}) \to \langle 1\rangle$$

### Theorem

*Let $n \geq 3$, $k \geq 1$, $h_1 < \ldots < h_k$, $m_1, \ldots, m_k$ be natural numbers. Let $\mathbb{Q}(\zeta_{m_i})$ be the field of $m_i$-th roots of unity and $\mathbb{Z}(\zeta_{m_i})$ its ring of integers. There is a subgroup $\Gamma \leq \mathrm{IA}(F_n)$ of finite index and a representation*

$$\rho : \Gamma \to \prod_{i=1}^{k} \mathrm{SL}((n-1)h_i, \mathbb{Q}(\zeta_{m_i}))^{m_i}$$

*such that $\rho(\Gamma)$ is commensurable with $\prod_{i=1}^{k} \mathrm{SL}((n-1)h_i, \mathbb{Z}(\zeta_{m_i}))^{m_i}$.*

Same Theorem for $\mathrm{Out}(F_n) := \mathrm{Aut}(F_n)/\mathrm{Inner\ Automorphisms}$

### Corollary

*The groups $\mathrm{IA}(F_3)$, $\mathrm{Out}(F_3)$ are large.*

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## Problem (SL)

- Is $\rho(\Gamma(G, \pi)) \leq \mathcal{G}^1_{G,\pi}$ (up to finite index)?

If not then Problem A is solved positively.

**Theorem**

*If $G$ is metabelian then $\rho(\Gamma(G, \pi)) \leq \mathcal{G}^1_{G,\pi}$ up to finite index.*

## Proof of General Theorem

$$\pi : F_n = \langle x_1, \ldots, x_{n-1},\ y \rangle \longrightarrow G \qquad (\pi(y) = 1)$$

Elements of $\Gamma(G, \pi)$

- $x_i \longrightarrow x_i y$, $x_j \longrightarrow x_j$ $(i \neq j)$, $y \longrightarrow y$;
- $x_i \longrightarrow y x_i$, $x_j \longrightarrow x_j$ $(i \neq j)$, $y \longrightarrow y$;
- $y \longrightarrow y \cdot \mathrm{relation}(x_1, \ldots, x_{n-1})$, $x_i \longrightarrow x_i$ (all $i$).

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Compute their action on $\bar{R}$.

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Compute their action on $\bar{R}$.

Use Theorems of Vaserstein which show that elementary matrices generate subgroups of finite index in $\mathrm{SL}((n-1)h_i, \mathcal{R}_i)$.