

# Majorana Calculations

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We present an alternative, slightly more general approach to the axiomatisation that happily allows us to find a universal Majorana algebra. It is work undertaken with S. Shpectorov.

# Axioms for the Algebra

## Part I of II

$M$  is a Majorana algebra over  $R$  if

1.  $R$  is a commutative, associative ring with 1 containing  $\mathbb{Q}$
2.  $M$  is an algebra over  $R$  with commutative, not necessarily associative product  $\cdot$
3.  $M$  admits a bilinear symmetric form  $\langle \cdot, \cdot \rangle$  that associates with  $\cdot$ , so  $\forall a, b, c \in M$ ,  $\langle a, b \cdot c \rangle = \langle a \cdot b, c \rangle$

and...

## Axioms for Majorana elements

...  $M$  is generated by marked Majorana elements  $m \in M$ :

- $m$  is an idempotent of length 1:  $m \cdot m = m$ ,  $\langle m, m \rangle = 1$
- the space  $m^\perp$  admits a direct sum decomposition into eigenspaces ( $\text{ad}_m : a \mapsto ma$ ,  $M_\mu^m = \{a \in M \mid ma = \mu a\}$ )

$$m^\perp = M_0^m \oplus M_{\frac{1}{4}}^m \oplus M_{\frac{1}{32}}^m$$

- and the eigenspaces obey the following fusion rules:

	1	0	1/4	1/32
1	1	0	1/4	1/32
0	0	0	1/4	1/32
1/4	1/4	1/4	1, 0	1/32
1/32	1/32	1/32	1/32	1, 0, 1/4

# Setup for Axioms 1 – 3

Fix  $n \geq 2$  for this section

- ▶ Let  $\hat{X}$  be the free commutative magma generated by  $\{\hat{m}_1, \dots, \hat{m}_n\}$ .
- ▶ Introduce the equivalence relation  $\sim$  on **pairs** of elements from  $\hat{X}$ , generated by  $(x, yz) \sim (xy, z)$ .
- ▶ Collection  $[\hat{X}, \hat{X}]$  of equivalence classes  $[u, v]$ .

# Axioms 1 & 2: the ring $\hat{R}$ , the algebra $\hat{M}$

- ▶ Let  $\lambda_{[u,v]}$  be an indeterminate for each  $[u, v]$
- ▶ and  $\hat{R} = \mathbb{Q}[\{\lambda_{[u,v]}\}_{[u,v] \in [\hat{X}, \hat{X}]}]$  the polynomial ring.

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- ▶ Let  $\hat{M} = \hat{R}\hat{X}$ : the free  $\hat{R}$ -module with basis  $\hat{X}$
- ▶ given multiplication induced linearly from the magma product.

# Why the equivalence relation?

For Axiom 3

- ▶ Endow  $\hat{M}$  with a bilinear form  $\langle, \rangle$  by setting  $\langle u, v \rangle = \lambda_{[u,v]}$  on the basis.
- ▶ It associates by construction:  
$$\langle x, y \cdot z \rangle = \lambda_{[x,yz]} = \lambda_{[xy,z]} = \langle x \cdot y, z \rangle.$$

# The Category

- ▶ Let  $\mathcal{C}$  be the category of algebras, generated by  $n$  marked elements, satisfying axioms 1 through 3.
- ▶ Morphisms are  $(\phi, \psi) : (R, M) \rightarrow (R', M')$  such that

$$\begin{aligned}\phi|_{\mathbb{Q}} &= \text{id}, & \psi|_R &= \phi, \\ \langle \psi(x), \psi(y) \rangle &= \phi(\langle x, y \rangle), \\ \psi &: \text{generators} \mapsto \text{generators}.\end{aligned}$$

- ▶  $\hat{M}$  is an initial object. Thus every other such algebra  $M$  has unique  $(\phi, \psi) : (\hat{R}, \hat{M}) \rightarrow (R, M)$ .

## Axiom 4: $m \cdot m = m$ , $\langle m, m \rangle = 1$

- ▶ For the marked generators  $\{m_1, \dots, m_n\}$  of  $M$  in  $\mathcal{C}$  to be Majorana elements, we express conditions using  $(\phi, \psi) : (\hat{R}, \hat{M}) \rightarrow (R, M)$ .
- ▶ Let  $\hat{m}$  be the generator such that  $\psi(\hat{m}) = m$ .  
If  $m \cdot m = m$ , then  $m \cdot m - m = 0$ , so  $\psi(\hat{m} \cdot \hat{m} - \hat{m}) = 0$ , so  $\hat{m} \cdot \hat{m} - \hat{m} \in \ker \psi$ .
- ▶ If  $\langle m, m \rangle = 1$ , then  $\langle \hat{m}, \hat{m} \rangle - 1 \in \ker \phi$ .

## Axioms 5, 6: eigenspaces

- ▶  $(\text{ad}_m : a \mapsto am) \in \text{End } M$   
so  $f \in \mathbb{Q}[t]$  maps  $\text{ad}_m$  to another endomorphism.
- ▶ For  $a \in m^\perp$ ,  $a = a_0 + a_{\frac{1}{4}} + a_{\frac{1}{32}}$ .  
Let  $f = t(t - \frac{1}{4})(t - \frac{1}{32})$ , so  $f(\text{ad}_m)(a) = 0$   
and  $f(\text{ad}_{\hat{m}})(\hat{a}) \in \ker \psi$  for all  $\hat{a} \in \hat{m}^\perp$ .

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and  $f(\text{ad}_{\hat{m}})(\hat{a}) \in \ker \psi$  for all  $\hat{a} \in \hat{m}^\perp$ .
- ▶ Set  $f_\mu$  such that  $f_\mu(\text{ad}_m)(a) = \alpha_\mu$  for  $\alpha_\mu$  a  $\mu$ -eigenvector
- ▶ Fusion rules: if  $M_\mu^m M_\nu^m \subseteq M_\kappa^m$  then

$$(\text{ad}_m - \kappa)(\alpha_\mu \cdot \beta_\nu) = 0 \quad \forall \mu, \nu\text{-eigenvectors}$$

$$\therefore (\text{ad}_m - \kappa)(f_\mu(a) \cdot f_\nu(b)) = 0 \quad \forall a, b \in M$$

$$\text{and } c \cdot (\text{ad}_m - \kappa)(f_\mu(a) \cdot f_\nu(b)) = 0 \quad \forall a, b, c \in M$$

so their preimages are in  $\ker \psi$ .

Each relation specifies elements that must be in  $\ker(\phi, \psi)$ .

**Observation.** If  $(\phi_i, \psi_i) : (\hat{R}, \hat{M}) \rightarrow (R_i, M_i)$  all satisfy the Majorana conditions for  $i \in I$ ,

then there exists  $(\phi, \psi)$  with  $\ker(\phi, \psi) = \bigcap_{i \in I} \ker(\phi_i, \psi_i)$ .

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**Theorem.** Majorana algebras with  $n$  marked generators form a subcategory  $\mathcal{M}$  of  $\mathcal{C}$  with an initial object  $M$ , the universal Majorana algebra with  $n$  marked generators.

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Sakuma's theorem: a classification of 2-generated Majorana algebras

# Binary Trees

Fix  $n = 2$

Let the Majorana elements and generators be  $m, p$ .

- ▶ Trees with one leaf:  $m$  and  $p$ .
- ▶ Trees, two leaves:  $[m, p]$
- ▶ Trees(3 leaves):  $[m, [m, p]]$  and  $[[m, p], p]$
- ▶  $\#T(4) = 5$
- ▶  $\#T(5) = 12$
- ▶  $\#T(\ell \geq 6) = 32, 86, 243, 696, 2045 \dots$

# Generating Trees

## Recursion!

- ▶ As an example:  $T(5)$  is  $T(4)$  with one extra leaf and  $T(3)$  with two extra leaves.
- ▶  $T(\ell) = \bigcup_{i=1}^{i < \frac{\ell}{2}} T(i) \times T(\ell - i)$

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- ▶  $T(\ell) = \bigcup_{i=1}^{i < \frac{\ell}{2}} T(i) \times T(\ell - i)$   
... if  $\ell$  is odd.
- ▶ If  $\ell$  is even, we need to step carefully around symmetry;  
 $T(\ell) = \Delta T(\frac{\ell}{2}) \cup \bigcup_{i=1}^{i < \frac{\ell}{2}} T(i) \times T(\ell - i)$   
where  $\Delta S$  is unordered  $S \times S$ .

# Equivalence Relation

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## Building the algebra

- ▶  $X$  the free commutative algebra on generators  $\{m, p\}$  is infinite.
- ▶ Fix some  $\ell$  and consider only relations involving trees with less than  $\ell$  leaves.
- ▶ Relations are e.g.  $(\text{ad}_m - \kappa)(f_\mu(a) \cdot f_\nu(b)) = 0$  and thus also  $c \cdot (\text{ad}_m - \kappa)(f_\mu(a) \cdot f_\nu(b)) = 0$  for trees  $a, b, c$ .
- ▶ Using the basis  $X$ , we get huge matrices to solve.



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