6-dimensional nilpotent Lie algebras

Csaba Schneider

Centro de Álgebra
Universidade de Lisboa

14 April 2011

with Willem de Graaf (Trento) e Serena Cicalò (CAUL)

Dimension $\leqslant 5$

There is a unique isomorphism class of nilpotent Lie algebras of dimension 1, and another class of dimension 2.

There are 2 nilpotent Lie algebras with dimension 3:

$$\begin{aligned} L_{3,1} &= \langle x, y, z \mid [x,y] = [x,z] = [y,z] = 0 \rangle\,; \\ L_{3,2} &= \langle x, y, z \mid [x,y] = z,\ [x,z] = [y,z] = 0 \rangle\,. \end{aligned}$$

…and 3 with dimension 4:

$$\begin{aligned} L_{4,1} &= \langle x, y, z, u \rangle = L_{3,1} \oplus \mathbb{F}; \\ L_{4,2} &= \langle x, y, z, u \mid [x,y] = z \rangle = L_{3,2} \oplus \mathbb{F}; \\ L_{4,3} &= \langle x, y, z, u \mid [x,y] = z,\ [x,z] = u \rangle\,. \end{aligned}$$

There are 9 Lie algebras with dimension 5.

Dimension 6

1. Morozov (1958): characteristic 0;
2. Gong (1998): over algebraically closed fields and $\mathbb{R}$;
3. de Graaf (2007): over fields of characteristic not 2.

The aim of our work was to revise de Graaf (2007) and to extend it to characteristic 2.

Our Theorem

Let $\mathbb{F}$ be a field and set $s = |\mathbb{F}^* : (\mathbb{F}^*)^2|$.

If $\operatorname{char} \mathbb{F} = 2$ then let t be the number of equivalence classes of the eq. relation $x \sim y$ iff $y = \alpha^2 x + \beta^2$ with $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$.

Theorem

(I) *If $\operatorname{char} \mathbb{F} \neq 2$, then the number of isomorphism classes of 6-dimensional nilpotent Lie algebras over $\mathbb{F}$ is $26 + 4s$.*

(II) *If $\operatorname{char} \mathbb{F} = 2$, then the number of isomorphism classes of 6-dimensional nilpotent Lie algebras over $\mathbb{F}$ is $30 + 2s + 4t$.*

For instance, there are

1. 34 algebras over $\mathbb{F}_q$ with q odd;
2. 30 over $\mathbb{C}$;
3. ∞ over $\mathbb{Q}$;
4. 36 over $\mathbb{F}_{2^n}$ or $\overline{\mathbb{F}_2}$.

The Skjelbred-Sund method (1979)

Let L and K be Lie algebras. Then K is a descendant of L if $Z(K) \leqslant K'$ and $K/Z(K) \cong L$.

Let L be a nilpotent Lie algebra over $\mathbb{F}$. Then either

(i) $L = L_1 \oplus \mathbb{F}$ or

(ii) L is a descendant of $L/Z(L)$.

Hence all nilpotent Lie algebras of dim 6 can be obtained by constructing

(i) the algebras $L \oplus \mathbb{F}$ where L is nilpotent and $\dim L = 5$;

(ii) 6-dim descendants of the nilp. Lie algebras with dim $\leqslant 5$.

How to the obtain descendants?

Theorem

Let L be a Lie algebra.

- *Then $H^2(L, \mathbb{F})$ is an $Aut(L)$-module.*
- *One-to-one correspondence between*
  1. *the set of isomorphism classes of the step-s descendants of L*
  2. *the $Aut(L)$-orbits on the set of s-dimensional allowable subspaces of $H^2(L, \mathbb{F})$.*

Lie algebra cohomology

Cocycle: An alternating bilinear form ϑ with the identity

$$\vartheta([x_1, x_2], x_3) + \vartheta([x_3, x_1], x_2) + \vartheta([x_2, x_3], x_1) = 0.$$

Coboundary: A cocyle of the form $\eta_\nu(x, y) = \nu([x, y])$ where ν is a linear form on L.

The sets of cocycles and coboundaries are denoted by $Z^2(L, \mathbb{F})$ and $B^2(L, \mathbb{F})$. Then $H^2(L, \mathbb{F}) = Z^2(L, \mathbb{F})/B^2(L, \mathbb{F})$.

A subspace $S \leqslant H^2(L, \mathbb{F})$ is allowable if

$$\bigcap_{\vartheta \in S} \vartheta^\perp \cap Z(L) = 0.$$

The Aut(L)-action: let $\vartheta \in Z^2(L, \mathbb{F})$ and $g \in \mathrm{Aut}(L)$; then

$$(g\vartheta)(x, y) = \vartheta(gx, gy).$$

GROUPS AND SEMIGROUPS: Interactions and Computations

Lisbon (Portugal) - July, 25-29, 2011

The aim of this conference is to deepen the existing interactions between group theory and semigroup theory.

The conference will be organized by the Centro de Álgebra da Universidade de Lisboa (CAUL), the Centro Internacional de Matemática (CIM), and the Departamento de Matemática da Faculdade de Ciências da Universidade de Lisboa (DM-FCUL)

INVITED SPEAKERS

Carlos André,	CELC, Universidade de Lisboa
Peter Cameron,	University of London
Bettina Eick,	Technical University Braunschweig
John Fountain,	University of York
Robert Gray,	CAUL, Universidade de Lisboa
Zur Izhakian,	Bar-Ilan University
Olga Kharlampovich,	McGill University
Michael Kinyon,	University of Denver
Charles Leedham-Green,	University of London
Alexander Lubotzky,	Hebrew University
Peter Neumann,	University of Oxford
Cheryl Praeger,	University of Western Australia
Ákos Seress,	University of Western Australia & Ohio State University
Pedro Silva,	CMUP, Universidade do Porto
Mikhail Volkov,	Ural State University
Efim Zelmanov,	University of California

ORGANIZING COMMITTEE

Gracinda M.S. Gomes (Chair), CAUL, DM-FCUL
João Araújo, CAUL, Universidade Aberta
Victoria Gould, University of York
Stephen Linton, University of St. Andrews
Eamonn O'Brien, University of Auckland
Csaba Schneider, CAUL, DM-FCUL

REGISTRATION AND CONTACTS

Further information is available at
http://caul.cii.fc.ul.pt/GSConf2011
To contact the organizers,
please send an email to
conf2011@cii.fc.ul.pt

A simple example

Let $L = L_{5,2} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3 \rangle$.

Define $\Delta_{i,j}$ as the alternating bilinear form on L by $\Delta_{i,j}(x_i, x_j) = -\Delta_{i,j}(x_j, x_i) = 1$ and $\Delta_{i,j}(x_k, x_l) = 0$ otherwise.

$$\begin{aligned} Z^2(L, \mathbb{F}) &= \left\langle \Delta_{1,2}, \Delta_{1,3}, \Delta_{1,4}, \Delta_{1,5}, \Delta_{2,3}, \Delta_{2,4}, \Delta_{2,5}, \Delta_{4,5} \right\rangle ; \\ B^2(L, \mathbb{F}) &= \left\langle \Delta_{1,2} \right\rangle ; \\ H^2(L, \mathbb{F}) &= \left\langle \overline{\Delta_{1,3}}, \overline{\Delta_{1,4}}, \overline{\Delta_{1,5}}, \overline{\Delta_{2,3}}, \overline{\Delta_{2,4}}, \overline{\Delta_{2,5}}, \overline{\Delta_{4,5}} \right\rangle . \end{aligned}$$

$$\mathrm{Aut}(L) = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & u & a_{34} & a_{35} \\ a_{41} & a_{42} & 0 & a_{44} & a_{45} \\ a_{51} & a_{52} & 0 & a_{54} & a_{55} \end{pmatrix} \tag{1}$$

where $u = a_{11}a_{22} - a_{12}a_{21}$.

A simple example

Let $S = \langle(\alpha_1, \ldots, \alpha_7)\rangle$ an allowable 1-space in $H^2(L, F)$.

Let B be the first automorphism if $\alpha_1 \neq 0$; the second otherwise

$$\begin{pmatrix} 1 & -\alpha_4\alpha_7 & 0 & 0 & 0 \\ 0 & \alpha_1\alpha_7 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1\alpha_7 & -\alpha_2 & -\alpha_3 \\ 0 & -(\alpha_1\alpha_6 - \alpha_3\alpha_4) & 0 & \alpha_1 & 0 \\ 0 & \alpha_1\alpha_5 - \alpha_2\alpha_4 & 0 & 0 & \alpha_1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -\alpha_4\alpha_7 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_4\alpha_7 & -\alpha_5 & -\alpha_6 \\ 0 & \alpha_3\alpha_4 & 0 & \alpha_4 & 0 \\ 0 & -\alpha_2 d & 0 & 0 & \alpha_4 \end{pmatrix}.$$

Then $BS = \langle(1, 0, 0, 0, 0, 0, 1)\rangle$.

The algebra L has only 1 descendant:

$$K = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3,\ [x_1, x_3] = [x_4, x_5] = x_6 \rangle .$$

A complicated example

Let $L = L_{4,1} = \langle x_1, \ldots, x_4 \rangle$.

$$\begin{aligned} Z^2(L,\mathbb{F}) &= \left\langle \Delta_{1,2}, \Delta_{1,3}, \Delta_{1,4}, \Delta_{2,3}, \Delta_{2,4}, \Delta_{3,4} \right\rangle ; \\ B^2(L,\mathbb{F}) &= 0 \\ H^2(L,\mathbb{F}) &= Z^2(L,\mathbb{F}) = \left\langle \Delta_{1,2}, \Delta_{1,3}, \Delta_{1,4}, \Delta_{2,3}, \Delta_{2,4}, \Delta_{3,4} \right\rangle . \end{aligned}$$

$\mathrm{Aut}(L) = \mathrm{GL}(4,\mathbb{F})$.
$H^2(L,\mathbb{F}) \cong L \wedge L$ as $\mathrm{Aut}(L)$-modules: $\Delta_{i,j} \mapsto x_i \wedge x_j$.
Let S_ε be the 2-space

$$S_\varepsilon = \langle (1,0,0,0,0,1), (0,1,0,0,\varepsilon,0) \rangle .$$

If $\mathrm{char}\ \mathbb{F} = 2$ let $\omega \in \mathbb{F} \setminus \{x^2 + x \mid x \in \mathbb{F}\}$ and define

$$R_\nu = \langle (1,0,0,0,0,1), (0,1,0,0,\nu,1) \rangle \text{ with } \nu \in \{0,\omega\}.$$

Theorem

- *In characteristic* $\neq 2$*:*
  - *Every orbit contains a subspace* S_ε *with some* $\varepsilon \in \mathbb{F}$*.*
  - *The subspaces* S_{ε_1} *and* S_{ε_2} *are in the same orbit iff* $\varepsilon_2 = \alpha^2 \varepsilon_1$ *with some* $\alpha \in \mathbb{F}^*$*.*

  In characteristic 2*:*
  - *In characteristic* 2 *every orbit contains a subspace* S_ε *or* R_ν *with some* $\varepsilon \in \mathbb{F}$ *and* $\nu \in \{0, \omega\}$*.*
  - *The subspaces* S_{ε_1} *and* S_{ε_2} *are in the same orbit iff* $\varepsilon_2 = \alpha^2 \varepsilon_1 + \beta^2$ *with some* $\alpha \in \mathbb{F}^*$ *and* $\beta \in \mathbb{F}$*.*
  - R_0 *and* R_ω *define additional orbits.*

Sketch of proof

- With some work it is easy to show that every orbit contains an element of the form S_ε or R_ν.
- Similarly, we may show that if ε_1 and ε_2 satisfy the corresponding properties then S_{ε_1} and S_{ε_2} are in the same orbit.

We use the fact that $\mathrm{GL}(4, \mathbb{F})$ preserves modulo scalars a quadratic form on H^2 with bilinear form f.

- In char $\neq 2$ we obtain that S_ε is a subspace with Gram-determinant ε wrt the given basis. This is used to separate the orbits of the S_ε.
- If $\mathrm{char}\ \mathbb{F} = 2$ then $f|_{S_\varepsilon}$ is identically zero while $f|_{R_\nu}$ is non-degenerate. The Arf-invariants are ε and ν, respectively. This is used to separate the orbits of the S_ε and R_ν.

Summary

Lie alg	char $\neq 2$	char $= 2$	Lie alg	char $\neq 2$	char $= 2$
$L_{3,1}$	1	1	$L_{5,3}$	2	2
$L_{3,2}$	0	0	$L_{5,4}$	0	0
$L_{4,1}$	$s+1$	$t+2$	$L_{5,5}$	1	2
$L_{4,2}$	$s+4$	$t+5$	$L_{5,6}$	2	$t+2$
$L_{4,3}$	1	1	$L_{5,7}$	3	$t+2$
$L_{5,1}$	0	0	$L_{5,8}$	$s+1$	$s+2$
$L_{5,2}$	1	1	$L_{5,9}$	s	$s+1$

Theorem

The number of 6-dimensional nilpotent Lie algebras over a field of char $\neq 2$ *is* $26+4s$ *while it is* $30+2s+4t$ *over char* 2*.*

Details are available

1. S. Cicalò, W. de Graaf, C. Schneider. Six-dimensional nilpotent Lie algebras. To appear in LAA, also on arxiv.org.
2. S. Cicalò, W. de Graaf, C. Schneider. LieAlgDB, a GAP4 package.