Our Groups

- connected reductive algebraic group G defined over the algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$ of characteristic q_0.
- F a Frobenius morphism of G
- $G^F = \{g \in G \mid F(g) = g\}$, finite group of Lie type, e.g. $G^F = \mathrm{SL}(d, q)$.

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Jordan Decomposition

Every $g \in G^F$ has unique *Jordan decomposition* $g = su = us$, where

- s semisimple (think diagonalisable),
- u unipotent (think all eigenvalues 1)

Note

- $o(s)$ co-prime to q_0
- $o(u)$ power of q_0

Quokka Sets

A *quokka*-set Q is a non-empty subset of G^F such that

a) If $g \in G^F$ has Jordan decomposition $g = su$ then $g \in Q \Leftrightarrow s \in Q$.

b) Q is a union of G^F-conjugacy classes.

Tori

A *torus* is an algebraic group isomorphic to

$$T \cong \overline{\mathbb{F}}_q^* \times \cdots \times \overline{\mathbb{F}}_q^*.$$

In particular, T is abelian.
T is *F-stable* if $F(T) = T$ and a *maximal torus* if T is closed and not properly contained in another torus.

Example: $G = \mathrm{SL}(d, \bar{\mathbb{F}}_q)$.

$T_0 =$ diagonal matrices in G.

The Weyl group W.

Choose T_0 an F-stable maximal torus in G;
Weyl Group $W = N_G(T_0)/T_0$

Example: $G = \mathrm{SL}(d, \bar{\mathbb{F}}_q)$.

$T_0 =$ diagonal matrices in G.
$W \cong S_d$

F-conjugacy

$v, w \in W$ are F-conjugate
if there is $x \in W$ such that

$$v = x^{-1} w F(x).$$

Example: $G = \mathrm{SL}(d, \bar{\mathbb{F}}_q)$.

F-conjugation is usual conjugation

A Correspondence

1-1 correspondence

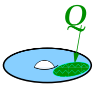
- G^F-conjugacy classes of F-stable maximal tori
- F-conjugacy classes of Weyl group.

Quokka-Tori

Q-*torus*	F-stable maximal torus T, s.t. $T \cap Q \neq \varnothing$
$\mathcal{T}_Q$	set of all Q-tori
Q-class	F-conjugacy class of W corresponding to some G^F-conjugacy class of Q-tori
$\mathcal{C}_Q$	set of all Q-classes
T_C	a representative element of $\mathcal{T}_Q$ corresponding to C for $C \in \mathcal{C}_Q$.

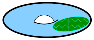
Quokka-Tori

Quokka-Tori



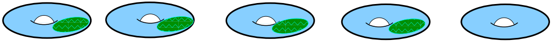
Quokka-Tori

Q–Torus

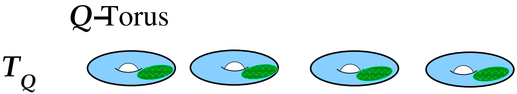


Quokka-Tori

Q–Torus



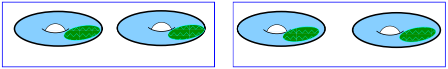
Quokka-Tori



Quokka-Tori

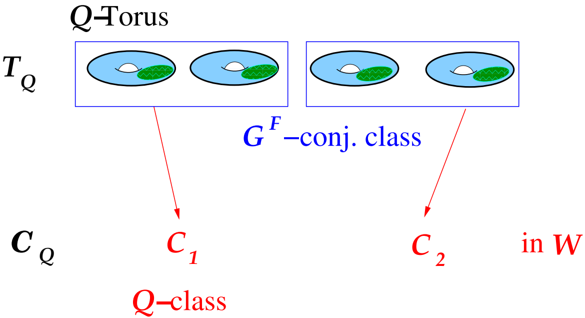
Q–Torus

T_Q

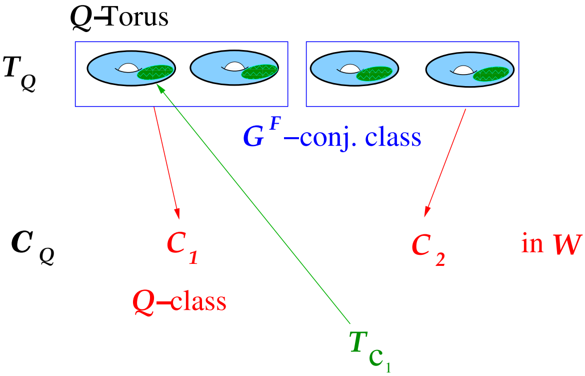


G^F–conj. class

Quokka-Tori



Quokka-Tori



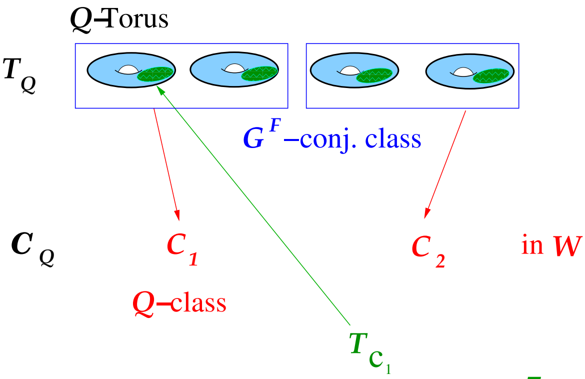
The Main Theorem

Quokka Theorem

Let G, F, T_0 and W be as above. Let $Q \subseteq G^F$ be a quokka set. Then

$$\frac{|Q|}{|G^F|} = \sum_{C \in \mathcal{C}_Q} \frac{|C|}{|W|} \cdot \frac{|T_C^F \cap Q|}{|T_C^F|}.$$

The Main Theorem



$$\frac{|Q|}{|G^F|} = \sum_{C \in \mathcal{C}_Q} \frac{|C|}{|W|} \cdot \frac{|T_C^F \cap Q|}{|T_C^F|}.$$

Bounds on proportions

In particular, if u_Q, ℓ_Q are positive constants such that $\ell_Q \leq \frac{|T^F \cap Q|}{|T^F|} \leq u_Q$ for all $T \in \mathcal{T}_Q$, and if $\hat{\mathcal{C}}_Q = \bigcup_{C \in \mathcal{C}_Q} C$, then,

$$\ell_Q \frac{|\hat{\mathcal{C}}_Q|}{|W|} \leq \frac{|Q|}{|G^F|} \leq u_Q \frac{|\hat{\mathcal{C}}_Q|}{|W|}.$$

Lower Bounds for $|Q|/|G^F|$

- Restrict to some $C \in C_Q$
- Find a (uniform) lower bound for $m_C = \frac{|T_C^F \cap Q|}{|T_C^F|}$ those C.
  Problem in abelian groups.
- Find lower bounds for $\sum_C \frac{|C|}{|W|}$ for those C.
  For classical groups: Problem in S_n or related groups.

Upper Bounds for $|Q|/|G^F|$

- Need to consider all $C \in C_Q$
- Find a (uniform) upper bound for $m_C = \frac{|T_C^F \cap Q|}{|T_C^F|}$ those C.
  Problem in abelian groups.
- Find upper bounds for $\sum_{C \in \mathcal{C}_Q} \frac{|C|}{|W|}$. For classical groups: Problem in S_n or related groups.