

Birman-Murakami-Wenzl algebras of Dynkin type

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joint work with
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Outline

- 1 Introduction
- 2 Kauffman tangle algebra
- 3 The Brauer algebra
- 4 About the proof
- 5 Other types than A_n

Definition

BMW = Birman-Murakami-Wenzl

Summary

The BMW algebra

- (i) of type A_{n-1} is isomorphic to the Kauffman algebra on n strands;

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The BMW algebra

- (i) of type A_{n-1} is isomorphic to the Kauffman algebra on n strands;
- (ii) of type D_n is isomorphic to the Kauffman algebra on n strands for an orbifold related to the cylinder;
- (iii) of type E_6 , E_7 , or E_8 reflects the combinatorial structure found in the above two series related to the Weyl groups;
- (iv) of type B_n should reflect some of the nice properties of the above series.

BMW algebra of simply laced type

Let M be a graph; adjacency \sim
and

$$R = \mathbb{Z}[l^{\pm 1}, m, \delta^{\pm 1}] / (m(1 - \delta) - (l - l^{-1}))$$

Definition

$BMW(M)$: Algebra over R

generators $\{g_i \mid i \in M\}$

relations

(braid1)	$g_i g_j = g_j g_i$	when $i \not\sim j$,
(braid2)	$g_i g_j g_i = g_j g_i g_j$	when $i \sim j$,
(skein)	$m e_i = l(g_i^2 + m g_i - 1)$	for all i ,
(self-intersection1)	$g_i e_i = l^{-1} e_i$	for all i ,
(self-intersection2)	$e_i g_j e_i = l e_j$	when $i \sim j$.

Further relations

$$\begin{array}{ll}
 (g_i - l^{-1})(g_i^2 + mg_i - 1) = 0 & \text{for all } i \\
 e_i^2 = \delta e_i & \text{for all } i \\
 g_i g_j e_i = e_j e_i & \text{whenever } i \sim j
 \end{array}$$

Similarly, a Matsumoto–Tits relation for each word

- $x_i y_j = y_j x_i$ (or smaller) with $i \neq j$, $i \not\sim j$
- $x_i y_j z_i = u_j v_i w_j$ (or smaller) with $i \sim j$

Properties of the BMW algebra

- **Opposition** is an anti-involution
- Modulo the ideal generated by the e_i , we obtain the Hecke algebra
- The subalgebra generated by the e_i is the Temperley–Lieb algebra
- Whenever Y is a coclique of M , then $e_Y := \prod_{i \in Y} e_i$ generates an ideal of BMW

Side remark on the Artin group $A(M)$

Definition

$A(M)$: Artin group

generators $\{s_i \mid i \in M\}$

relations

(braid1) $s_i s_j = s_j s_i$ when $i \not\sim j$,

(braid2) $s_i s_j s_i = s_j s_i s_j$ when $i \sim j$

Theorem (Krammer,...)

If M is spherical then $A(M)$ embeds in $BMW(M)$ via $s_i \mapsto g_i$.

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BMW(A_{n-1}) and Kauffman tangles

Theorem (Morton-Wasserman)

The BMW algebra of type A_{n-1} is isomorphic to the Kauffman tangle algebra of (n, n) -tangles.

Tangles

- An (n, k) -tangle is a rectangular slice of a diagram with n strands entering at the top and k strands ending at the bottom.

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- An (n, k) -tangle is a rectangular slice of a diagram with n strands entering at the top and k strands ending at the bottom.
- Composition of tangles turns the set of (n, n) -tangles into a ring.

Definition of Kauffman tangle algebra

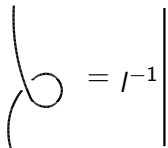
The Kauffman tangle algebra over R consists of R -linear combinations of tangles modulo

- **regular isotopy** (planar isotopy and Reidemeister moves II, III)
- **self-intersection**: factor l
- **disjoint loops**: factor δ
- **skein relation**: parameter m

Composition gives algebra multiplication.

$$R = \mathbb{Z}[\delta^{\pm 1}, m, l^{\pm 1}] / (m(1 - \delta) - (l - l^{-1}))$$

Definition of self-intersection



The diagram shows a single strand on the left that crosses itself once, forming a loop. This is followed by an equals sign and the expression l^{-1} multiplied by a vertical straight strand.

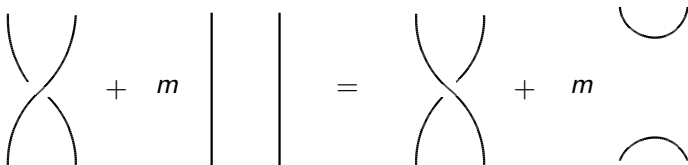
Straightening at a cost of l^{-1} .
Other orientation at a cost of l .

Definition of disjoint loop

$$\bigcirc = \delta$$

Removing a disjoint loop at cost of δ .

The skein relation



The diagram illustrates the skein relation in the Kauffman tangle algebra. It shows an equation where the left side is the sum of a crossing and m times a vertical line, and the right side is the sum of a crossing and m times a pair of arcs (one top and one bottom).

$$\text{Crossing} + m \text{ (vertical line)} = \text{Crossing} + m \text{ (two arcs)}$$

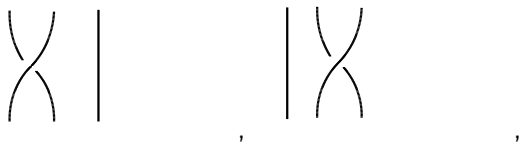
Side remark on Kauffman bracket

The Kauffman algebra on $(0,0)$ -tangles gives a link invariant in R , after correction for [writhe](#).

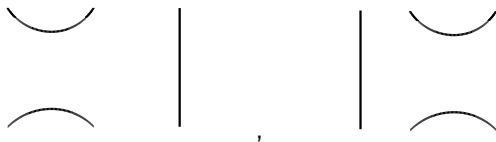
[writhe](#) = number of oriented crossings

The isomorphism from BMW to Kauffman tangles

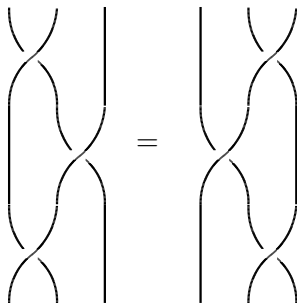
g_1 and g_2 are mapped to



e_1 and e_2 are mapped to

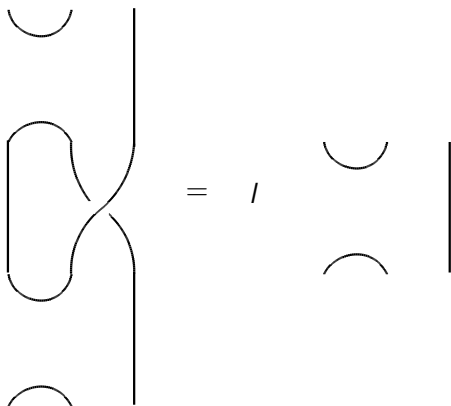


Example: Reidemeister III



$$g_1 g_2 g_1 = g_2 g_1 g_2$$

Example: self-intersection



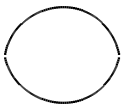
corresponds to $e_1 g_2 e_1 = I_{e_1}$

Example: skein relation

The diagram illustrates a skein relation in the Kauffman tangle algebra. On the left side of the equation, there is a crossing of two strands (a positive crossing) plus a term m multiplied by two parallel vertical strands. This is equal to the right side, which is a crossing of two strands (a negative crossing) plus a term m multiplied by two separate semi-circular arcs, one above the other.

$$g_i + m = g_i^{-1} + me_i$$

Example: disjoint loop



=

δ



$$e_i^2 = \delta e_i$$

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Definition of Brauer algebra of type M

Over $\mathbb{Z}[\delta^{\pm 1}]$ with generators $\{r_i, e_i \mid i \in M\}$ and relations

(involution)	$r_i^2 = 1$	for all i ,
(braid1)	$r_i r_j = r_j r_i$	when $i \not\sim j$,
(braid2)	$r_i r_j r_i = r_j r_i r_j$	when $i \sim j$,
(self-intersection1)	$r_i e_i = e_i r_i = e_i$	for all i ,
(self-intersection2)	$e_i r_j e_i = l e_j$	when $i \sim j$.
(commuting1)	$r_i e_j = e_j r_i$	when $i \not\sim j$,
(braid3)	$r_j e_i r_j = r_i e_j r_i$	when $i \sim j$,
(braid4)	$r_j r_i e_j = e_i r_j r_i = e_i e_j$	when $i \sim j$,
(idempotent)	$e_i^2 = \delta e_i$	for all i ,
(commuting2)	$e_i e_j = e_j e_i$	when $i \not\sim j$,
(braid5)	$e_i e_j e_i = e_i$	when $i \sim j$

Properties of the Brauer algebra of type M

- Opposition is anti-involution
- $W = W(M)$, the Coxeter group of type M , is a subgroup
- The e_i generate the Temperley–Lieb algebra of type M
- Whenever Y is a coclique of M , then $e_Y := \prod_{i \in Y} e_i$ generates an ideal of the Brauer algebra

From BMW to Brauer

Fix type M .

There is a ring homomorphism $\text{BMW} \rightarrow \text{Brauer algebra}$

- Coefficients from R specialize $l \mapsto 1, m \mapsto 0$
- $g_i \mapsto r_i$ and $e_i \mapsto e_i$
- surjective

Brauer algebra of type A_{n-1}

- An (n, k) -Brauer diagram is obtained from a tangle by forgetting whether a crossing is over or under

$n!! =$ the number of matchings of $2n$ points $= 1 \cdot 3 \cdots (2n - 1)$

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- $BW(A_{n-1})$ is free of rank $n!!$

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Rank and freeness results

$BMW(M)$ has finite rank

\Leftrightarrow

connected components of M are in $\{A_n, D_n, E_6, E_7, E_8 \mid n \in \mathbb{N}\}$.

$$\dim BMW(A_n) = (n+1)!!$$

$$\dim BMW(D_n) = (2^n + 1)n!! - (2^{n-1} + 1)n!$$

$$\dim BMW(E_6) = 1,440,585$$

$$\dim BMW(E_7) = 439,670,025$$

$$\dim BMW(E_8) = 53,328,069,225$$

Theorem

Let M be a Coxeter diagram of type A_n ($n \geq 1$), D_n ($n \geq 4$), or E_n ($6 \leq n \leq 8$).

- (i) *The BMW algebra of type M over R is free of the same rank as the Brauer algebra of type M .*
- (ii) *When tensored with $\mathbb{Q}(I, \delta)$, it is semisimple.*
- (iii) *It is cellular provided the coefficient ring R is extended to an integral domain containing inverses of all bad primes.*

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Notions needed for Brauer algebra analysis

- Root systems of type M

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- Minimal elements a_B for $B \in WB_Y$

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- A set \mathcal{Y} of cocliques such that B_Y for $Y \in \mathcal{Y}$ are W -orbit representatives
- Minimal elements a_B for $B \in WB_Y$
- Subgroup H_Y centralizing e_Y in Brauer algebra that is isomorphic to a Coxeter group (of type C_Y)

Admissible sets

\mathcal{A} is a collection of sets of orthogonal positive roots.

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$$r_i B = \{\pm r_i \alpha \mid \alpha \in B\} \cap \Phi^+$$

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- Can choose a set \mathcal{Y} of cocliques Y of M such that each W -orbit in \mathcal{A} has a representative $B_Y = \{\alpha_i \mid i \in Y\}^{\text{cl}}$

Brauer monoid action

Extend given W -action on \mathcal{A} to Brauer monoid by

$$e_i B = \begin{cases} B & \text{if } \alpha_i \in B \\ (B \cup \{\alpha_i\})^{\text{cl}} & \text{if } \alpha_i \perp B \\ r_\beta r_i B & \text{if } \beta \in B \setminus \alpha_i^\perp \end{cases}$$

Opposition is an anti-involution $x \mapsto x^{\text{op}}$.

It gives a right action on \mathcal{A} .

Centralizer of e_γ

- There is a height function on \mathcal{A} such that B_γ has height 0

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- Let C_γ be the set of $i \in M$ such that α_i is orthogonal to the highest element of WB_γ .

Centralizer of e_Y

- There is a height function on \mathcal{A} such that B_Y has height 0
- Let C_Y be the set of $i \in M$ such that α_i is orthogonal to the highest element of WB_Y .
- Then the centralizer of e_Y in the Brauer monoid contains a Coxeter group H_Y of type C_Y with unit $\delta^{-|Y|}e_Y$.

Monoid action in case of type A_{n-1}

$$\Phi^+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n\}$$

$$\mathcal{Y} = \{\emptyset, \{1\}, \{1, 3\}, \{1, 3, 5\}, \dots\}$$

- left action on bottom strands
- right action on top strands
- strand $\{i, j\} \leftrightarrow$ root $\epsilon_i - \epsilon_j$
- $Y = \{1, 3, 5, \dots, 2p - 1\}$. Then $C_Y = A_{n-2p-1}$, so $H_Y \cong \text{Sym}_{n-2p}$.

Nonempty $Y \in \mathcal{Y}$

M	$ B_Y $	B_Y^\perp	C_Y	Y
E_6	1	A_5	A_5	6
E_6	2	A_3	A_2	4, 6
E_6	4	\emptyset	\emptyset	2, 3, 5
E_7	1	D_6	D_6	7
E_7	2	$A_1 D_4$	$A_1 A_3$	5, 7
E_7	3	D_4	A_2	2, 5, 7
E_7	4	A_1^3	A_1	2, 3, 5
E_7	7	\emptyset	\emptyset	2, 3, 5, 7
E_8	1	E_7	E_7	8
E_8	2	D_6	A_5	6, 8
E_8	4	D_4	A_2	2, 3, 5
E_8	8	\emptyset	\emptyset	2, 3, 5, 7

Minimal elements a_B

Lemma

Let $Y \in \mathcal{Y}$. For each $B \in WB_Y$ there is a unique element a_B of minimal height (number of r_i) such that $B = a_B B_Y$.

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- Establish a rewriting system for Brauer monomials extending Matsumoto–Tits
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- Establish a rewriting system for Brauer monomials extending Matsumoto–Tits
- The rank of BMW algebra is at least the rank of the Brauer algebra. It will be bounded from above by this rank if the rewriting system reduces each monomial to a unique one from a fixed basis of the Brauer algebra.
- The Brauer monoid has a left and a right action on a collection \mathcal{A} of admissible sets of roots.
- The rewriting system has reduced elements of the form $a_B e_Y z a_C^{\text{op}}$ with B and C in the W -orbit of B_Y and $z \in H_Y$.

Representations

Lemma

$$\text{rank}(BMW(M)) = \sum_{Y \in \mathcal{Y}} |WB_Y|^2 \cdot |W(C_Y)|.$$

Theorem

sum of matrix algebras and has rank

$$\text{rank}(BMW(A_{n-1})) = n!! = \sum_{p=0}^{\lfloor n/2 \rfloor} \left(\frac{n!}{2^p p! (n-2p)!} \right)^2 \cdot (n-2p)!$$

$n!! =$ *the number of matchings of $2n$ points* $= 1 \cdot 3 \cdots (2n-1)$

The Brauer rewrite system

label	relation	label	relation
for i			
(R)	$r_i^2 \rightsquigarrow 1$	(R)	$e_i r_i \rightsquigarrow e_i$
(R)	$r_i e_i \rightsquigarrow e_i$	(H)	$e_i^2 \rightsquigarrow \delta e_i$
for $i \neq j$			
(H)	$r_i r_j \rightsquigarrow r_j r_i$	(H)	$e_i r_j \rightsquigarrow r_j e_i$
(H)	$e_i e_j \rightsquigarrow e_j e_i$		
for $i \sim j$			
(H)	$r_i r_j r_i \rightsquigarrow r_j r_i r_j$	(H)	$r_j e_i r_j \rightsquigarrow r_i e_j r_i$
(R)	$r_j r_i e_j \rightsquigarrow e_i e_j$	(R)	$e_i r_j r_i \rightsquigarrow e_i e_j$
(H)	$r_j e_i e_j \rightsquigarrow r_i e_j$	(R)	$e_i r_j e_i \rightsquigarrow e_i$
(H)	$e_j e_i r_j \rightsquigarrow e_j r_i$	(H)	$e_i e_j e_i \rightsquigarrow e_i$
for $i \sim j \sim k$			
(H)	$e_j e_i r_k e_j \rightsquigarrow e_j r_i e_k e_j$	(R)	$e_j r_i r_k e_j \rightsquigarrow e_j e_i e_k e_j$

Rewrite rules

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Rewrite rules

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- (H) means Homogeneous: same number of r_i left and right
- (R) means Reducing: fewer r_i right than left
- Work with words modulo the homogeneous relations
- Extend Matsumoto–Tits
- Reduce number of r_i in a word. (0: Temperley–Lieb)

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Type D_n

Similar to A_n case (but more complicated).

Pole of order two.

Type E_n

No diagrams

Ad hoc rewrite verification. (GAP used)

computational aspects

- GBNP rank computations
- Proof uses GAP rewriting in cases for type E
- Translation of monomials back and forth to triples of two admissible sets and a Weyl group element allow for computational model of the algebra

Temperley-Lieb algebras

Subalgebra of Brauer or BMW algebra generated by e_i

- Graham's definition: Hecke algebra modulo longest element relation for each rank 2 parabolic. [Coxeter type](#)

Temperley-Lieb algebras

Subalgebra of Brauer or BMW algebra generated by e_i

- Graham's definition: Hecke algebra modulo longest element relation for each rank 2 parabolic. [Coxeter type](#)
- Fan, Green's definition: B_n as fixed points of involution on A_{2n-1} . Differs from definition of C_n as fixed points of involution on D_{n+1} . [Dynkin type](#)

The rank of $BMW(B_n)$

$$a_{2n} = \sum_{i=0}^n \left(\sum_{p_0+2p_1=i} \frac{n!}{p_0!p_1!(n-p_0-2p_1)!} \right)^2 2^{n-i} (n-i)!$$

$$a_n = a_{n-1} + 2(n-1)a_{n-2}$$

$$a_0 = a_1 = 1$$

BMW algebras are of interest in their own right
Still I hope they will shed more light
On a particular quantum group category
Whose existence is yet another story

Thank you

