Computing Mapping Class Orbits

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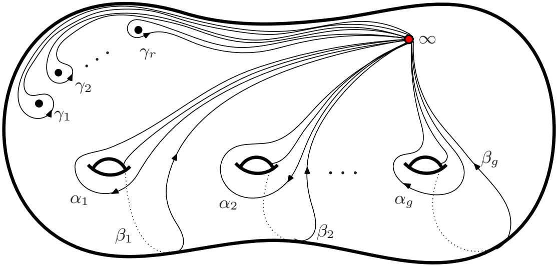
The Fundamental group

The fundamental group of a surface of genus g, with r punctures, is generated by the following loops. It also has the following presentation.

$$\langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots \gamma_r \mid [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \gamma_1 \cdots \gamma_r \rangle$$



The Mapping Class Group

Definition

The mapping class group of a surface is the group of homeomorphisms of a surface, up to isotopy/homotopy.

$$\mathrm{Mod}(S) = \mathrm{Homeo}^+(S)/\mathrm{Homeo}_0(S)$$

Dehn Twists

Definition

Given a curve α, a regular neighbourhood, N, of α and a homeomorphism $\varphi : N \to A$, where A is the annulus defined by

$$A = \{x \in \mathbb{C} \mid 1 \leq |x| \leq 2\}$$

Define the map $\theta : A \to A$ by

$$\tau_\alpha(z = re^{i\theta}) = re^{(i(\theta+(r-1)2\pi)}$$

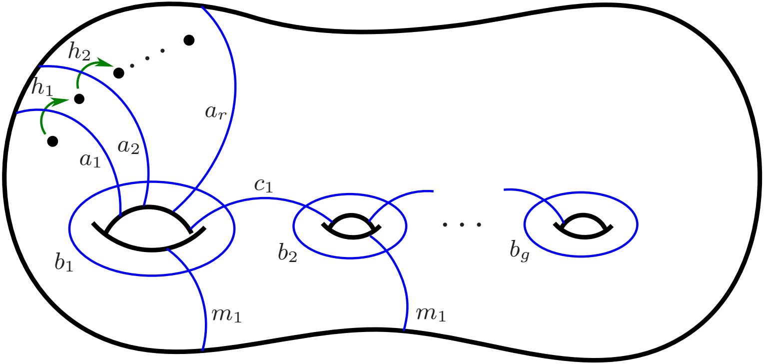
Then the *Dehn twist* about α is the map which is identity on all but N and on N is given by

$$\varphi^{-1} \circ \theta \circ \varphi$$

Dehn Twists

Theorem (Lickorish)

The mapping class group is generated by *Dehn twists*.



Dehn-Nielsen-Baer Theorem

Theorem (Baer,Dehn,Nielsen)

$$\mathbf{Mod}^{\pm}(S) \cong \mathbf{Out}(\pi_1(S, p))$$

Automorphism Groups of Riemann Surfaces

Theorem (Hurwitz)

Let X be a Riemann surface of genus $g \geq 2$, then

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Theorem (Hurwitz 1893)

Any finite group is isomorphic to a group of automorphisms of some compact Riemann surface (of genus at least 2).

Surface and Group Pairs

We say that the two surface-group pairs $[X_1, G_1]$ and $[X_2, G_2]$ are equivalent if their exists a biholomorphic map $\phi : X_1 \to X_2$ and group isomorphism $\theta : G_1 \to G_2$ such that for all $g \in G_1$, $\theta \circ g = \phi(g) \circ \theta$.

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This is equivalent to the existence of orientation preserving biholomorphic homeomorphisms $\theta : X_1 \to X_2$ and $\theta_0 : X_1/G_1 \to X_2/G_2$ such that $\pi_2 \circ \theta = \theta_0 \circ \pi_1$

Ramification Type

We say a Riemann surface with a group of automorphisms G is of *ramification type* $(g, G, \mathbf{C} = (C_1, \ldots, C_r))$ if X has genus g and the Riemann surface $Y = X/G$ is ramified over r points $\{y_1, \ldots, y_r\}$, which can be ordered in such a way so that the if $\pi(x_i) = y_i$ then the inertia subgroup G_{x_i} has distinguised generator g_i and $C_i = g_i^G$. By removing the branch points and their pre-images, the quotient map is a covering space, with deck transformation group isomorphic to G.

Hurwitz space

We denote the set equivalence clases of Riemann surfaces of ramification type $(g, G, \mathbf{C})$ by $\mathcal{H}(g, G, \mathbf{C})$. The Riemann-Hurwitz formula says that

$$2g - 2 = |G|[2g(X/G) - 2 + \sum_i 1 - \frac{1}{|C_i|}]$$

So the ramification data determines the genus g_0 of the base space X/G.

The Moduli Space of Riemann surfaces

Write $\mathcal{M}_g$ for the space of all Riemann surfaces of genus g.

There is the obvious map $\psi : \mathcal{H}(g, G, \mathbf{C}) \to \mathcal{M}_g$ we get by forgetting about the group. We denote the image of this map by $\mathcal{M}(g, G, \mathbf{C})$.

How many components does $\mathcal{M}(g, G, \mathbf{C})$ have?

The Group Theoretical Version

Let $\mathcal{E}$ be the set of tuples $(a_1, \ldots, a_{g_0}, b_1, \ldots, b_g, c_1, \ldots, c_r) \in G^{2g_0+r}$ such that:

- $a_1, \ldots, a_{g_0}, b_1, \ldots, b_{g_0}, c_1, \ldots, c_r$ generate G
- $c_i \in C_i$
- $\prod_{i=1}^{g_0}[a_i, b_i] \prod_{j=1}^{r} g_j = 1$

These tuples are taken up to conjugation in G

Theorem

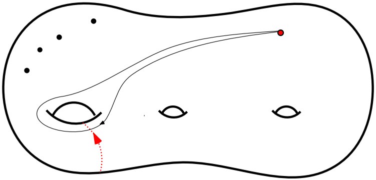
The following are in one-to-one correspondence:

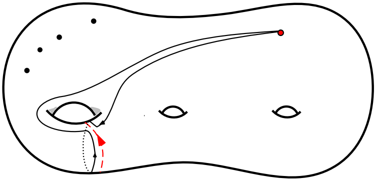
$$\left\{ \begin{array}{c} \text{Components of} \\ \mathcal{M}(g, G, \mathbf{C}) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Mapping Class Orbits of} \\ \mathcal{E} \end{array} \right\}$$

What are Mapping Class Orbits?

Where does this mysterious $\mathcal{E}$ come from?

The tuples correspond to surjective homomorphisms $\pi_1(S_{g_0,r}, *) \to G$. The loops around the branch points must be mapped to the specified conjugacy classes.





$$\alpha \mapsto \beta_1^{-1}\alpha_1$$

$$(\boldsymbol{a}_1, \ldots, \boldsymbol{c}_r) \mapsto (\boldsymbol{b}_1^{-1}\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots \boldsymbol{c}_g)$$

The MAPCLASS Package for GAP

- Based on the BRAID package (2002)
- Expanded to cover arbitrary genus
- Two main routines:
  - `GeneratingMCOrbits(group, genus, tuple)`
  - `AllMCOrbits(group, genus, tuple)`

How do we compute these orbits?

- Compute the structure constant. (`ClassStructCharConst`)
- Choose a random tuple and compute its orbit.
- Apply weighting to get small orbits.

Computational Issues and Possible Solutions

- The orbits are large and orbit size grows rapidly with genus.
- Computing the structure constant for long tuples is slow.

A possible solution to this is a splitting procedure.

Genus 0

- In Genus 0 then the mapping class group is just the braid group.
- The action is generated by braid twists

$$(g_1, \ldots, g_i, g_{i+1}, \ldots, g_r) \to (g_1, \ldots, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, \ldots, g_r)$$

- J. Wang has a working version of the splitting process.

The Splitting Process

- Take a tuple $(g_1, \ldots, g_{2r})$
- Split into two $(g_1, \ldots, g_r, x)$ and $(x^{-1}, g_{r+1}, \ldots g_{2r})$
- Calculate the orbits of these two halves, $\mathcal{O}_1$ and $\mathcal{O}_2$ as x runs over conjugacy classes.
- A triple $(\mathcal{O}_1, h, \mathcal{O}_2)$ is called a matching pair if $h \in C_G(x)$ and $\langle \mathcal{O}_1^h, \mathcal{O}_2 \rangle = G$.

The Splitting Process: Part 2

- We form a graph whose vertices are matching pairs.
- We join two matching pairs $(\mathcal{O}_1, h, \mathcal{O}_2)$, and $(\mathcal{O}_1', h', \mathcal{O}_2')$ if for any tuple $t \in (\mathcal{O}_1, h, \mathcal{O}_2)$ there is a pure braid taking t to $t' \in (\mathcal{O}_1', h', \mathcal{O}_2')$.
- Only works with one orbit.

An Example

- Want to compute the braid orbit for A_8 with tuple of eight involutions.
- The original package BRAID cannot compute this (10^{14} tuples).
- BRAID can do tuple of 5 elements in approximately 20 minutes.
- Get a graph with approximately 30,000 vertices.
- Connecting the graph takes approximately 3 hours.
- One orbit.

Further Work

- Generalize the splitting procedure for arbitrary genus.
- Parallel computation.
- Find invariants for orbits.