MA343 Group Theory I – Assignment 2

October 18, 2012, Lecturer: Claas Röver

QUESTION 1. Suppose $\alpha\colon G \to H$ is a surjective homomorphism of groups. Verify the following claims, where $g, \bar{g} \in G$ and $U \subset G$ is a subgroup.

(a) The order of g^α divides the order of g. (Assume that every integer divides infinity.)
(b) If g and $\bar{g}$ commute, then so do g^α and $\bar{g}^\alpha$.
(c) The image of U, i.e. U^α, is a subgroup of H, and if U is normal in G, then U^α is normal in H.

QUESTION 2. Suppose U and V are subgroups of a group G.

(a) Show that $U \cap V$ is also a subgroup of G.
(b) Show that if V is normal in G, then $UV = \{uv \mid u \in U, v \in V\}$ is a subgroup of G.
(c) Find an example to show that UV is not a subgroup in general.

QUESTION 3. (a) Let $D = G \times H$ be the direct product of groups G and H. Prove that D has a normal subgroup N, such that $N \cong G$ and $D/N \cong H$.

(b) Let U and V be subgroups of a group G. Then the following are equivalent.
(i) $G \cong U \times V$ **(ii)** $[U, V] = 1$, $U \cap V = 1$ and $UV = G$.

QUESTION 4. Suppose that G is a group with $g^2 = 1$ for all $g \in G$. Show that G is necessarily abelian. Prove that if G is finite, then $|G| = 2^k$ for some $k \geq 0$ and G needs at least k generators.

QUESTION 5. Consider S_5, the symmetric group of degree five. Does it have a subgroup isomorphic to $C_5 \times C_5$? Does it have elements of order 6? Does it have a subgroup isomorphic to D_5? What about a subgroup isomorphic to D_6?

Hand in your solutions at the **beginning** of the lecture on **Friday, October 26, 2012.**