MA343 Group Theory I – Problem Sheet 1

September 14, 2012, Lecturer: Claas Röver

QUESTION 1. Show that every element of a group has precisely one inverse.

QUESTION 2. Decide whether $(\mathbb{Z}, -)$, i.e. the integers with subtraction as binary operation, is a group. What about the integers modulo m with multiplication $(\mathbb{Z}/m\mathbb{Z}, \cdot)$, with $m \in \mathbb{Z}$?

QUESTION 3. (a) Find a group with four elements in which every element is its own inverse.

(b) Find a group with four elements in which not every element is its own inverse.

QUESTION 4. Let $a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$; 2×2 matrices with complex entries.

(a) Describe the smallest group of 2×2 complex matrices containing b.

(b) Describe the smallest group of 2×2 complex matrices containing a.

QUESTION 5. Let $\mathcal{S}$ be the square with vertices $A = (1, 1)$, $B = (-1, 1)$, $C = (-1, -1)$, $D = (1, -1) \in \mathbb{R}^2$. As we know, a square has eight symmetries. Find the eight corresponding 2×2 matrices (with real entries). Then verify that the composition of a reflection in a diagonal of the square and a reflection which keeps no vertex fixed is a rotation about the origin by 90^o.

QUESTION 6. Show that the complex numbers of modulus one, i.e. $\{z \in \mathbb{C} \mid |z| = 1\}$, with multiplication form a group. Then find a symmetric object which has this group as a group of, not necessarily all, its symmetries.