Groups of order 8

We need a few facts.

① A finite abelian group is a direct product of cyclic groups.

② If a finite group G has order p^k for a prime p, then the centre of G is non-trivial. (See Semester II)

③ If $G/Z(G)$ is cyclic, then $G = Z(G)$.

Proof: Let $Z = Z(G)$. Since $G/Z(G)$ is cyclic, there exists $x \in G$ with $G/Z(G) \cong \langle Zx \rangle$, and every element $g \in G$ is of the form zx^i for some $z \in Z$ and $i \in \mathbb{Z}$. But then

$$z_1 x^{i_1} z_2 x^{i_2} = z_1 z_2 x^{i_1+i_2} = z_2 z_1 x^{i_2+i_1} = z_2 x^{i_2} z_1 x^{i_1}$$

and so G is abelian, i.e. $Z(G) = G$. $\square$

Now suppose $|G| = 2^3 = 8$, and put $Z = Z(G)$. There are two cases

I. $Z = G$, i.e. G is abelian and hence one of

$$C_8, \quad C_4 \times C_2 \quad \text{or} \quad C_2 \times C_2 \times C_2$$

II. $|Z| = 2$ and $G/Z \cong C_2 \times C_2$ and G is not abelian!

Pick a and $b \in G$ s.t. $G/Z = \langle Za, Zb \rangle$. Let $Z = \langle z \rangle$. So $z^2 = 1$ and $a^2, b^2, (ab)^2 \in Z$ and $G = \langle a, b, z \rangle$. Since G is not abelian, but G/Z is abelian, we get $z = [a,b] = a^{-1}b^{-1}ab$.

(i) $a^2 = b^2 = 1$. Then $[a,b] = (ab)^2 = z$. So $r = ab$ has order 4, and

$$G \cong D_4 \quad \text{(generated by } a \text{ and } b\text{)}.$$

(ii) $a^2 = 1$ and $b^2 = z$. Then $[a,b] = ab^{-1}ab = b^2$, i.e. $(ab^{-1})^2 = 1$.

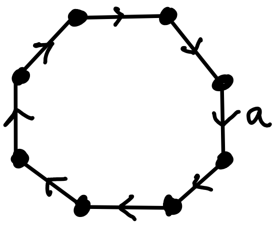
$$\text{So } G = \langle a, ab^{-1} \mid a^2 = (ab^{-1})^2 = 1, (aab^{-1})^4 \rangle \cong D_4.$$

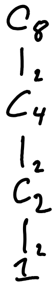
(iii) $a^2 = z$ and $b^2 = 1$ also yields $G \cong D_4$.

(iv) $a^2 = z$ and $b^2 = z$. Then $a^{-1} = a^3 = za$ and $b^{-1} = zb$, so $[a,b] = zazbab = (ab)^2 = z$. Putting $c = ba$, we find $abc = 1$ and $c^2 = baba = a^{-1}(ab)^2 a = z$. Reading z as -1 it is now easy to check that $G \cong Q_8$.

Cayley Graphs and Subgroups of Groups of Order Eight

$C_8 = \langle a \mid a^8 = 1 \rangle$

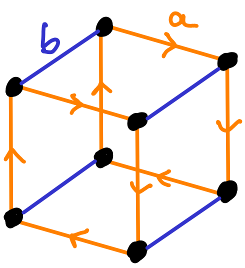


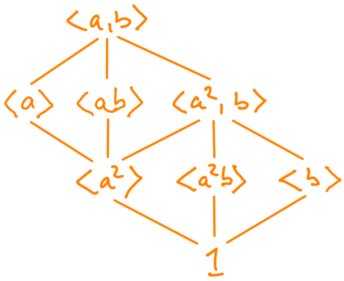


order	no. of elemts
8	4
4	2
2	1
1	1

one generator,
8 conjugacy classes

$C_4 \times C_2 = \langle a, b \mid a^4 = b^2 = 1, ab = ba \rangle$ two generators,

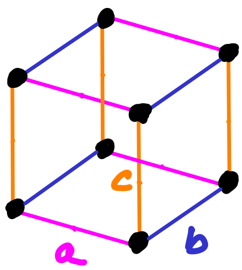


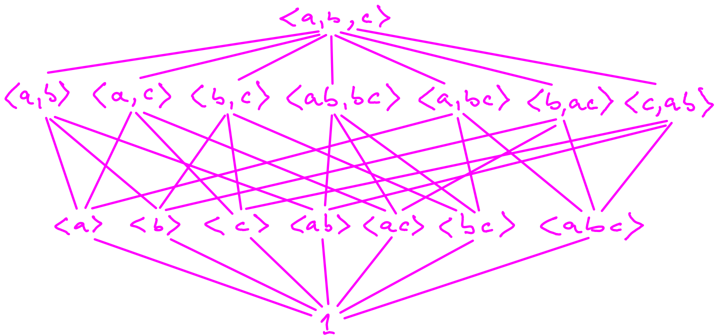


order	no. of elemts
4	4
2	3
1	1

8 conjugacy classes

$C_2 \times C_2 \times C_2 = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^2 = (bc)^2 = (ac)^2 = 1 \rangle$

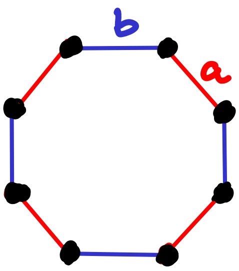


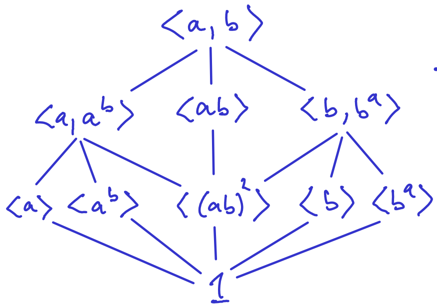


three generators
8 conjugacy classes

order	no. of elemts
2	7
1	1

$D_4 = \langle a, b \mid a^2 = b^2 = 1 = (ab)^4 \rangle$ two generators



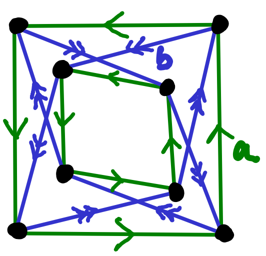


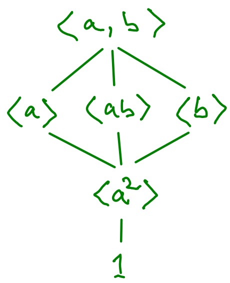
order	no. of elemts
4	2
2	5
1	1

5 conjugacy classes:
$\{1\}$, $\{ab, ba\}$, $\{(ab)^2\}$,
$\{a, bab\}$, $\{b, aba\}$

Centre: $\langle (ab)^2 \rangle$

$Q_8 = \langle a, b \mid a^2 = b^2, a^4 = 1, a^{-1}ba = b^{-1} \rangle$ two generators





order	no. of elemts
4	6
2	1
1	1

5 conjugacy classes:
$\{1\}$, $\{a^2\}$, $\{a, a^{-1}\}$, $\{b, b^{-1}\}$,
$\{ab, (ab)^{-1}\}$

Centre: $\langle a^2 \rangle = \langle (ab)^2 \rangle$