Solution to Question 4 on Sheet 5 for MA286

Given $f: \mathbb{R} \to \mathbb{R}$ which is differentiable, then f has its extreme points at those x_c for which $f'(x_c) = 0$. At those points the formula for the curvature from Question 3 becomes

$$\kappa(x_c) = \frac{|f''(x_c)|}{(1+f'(x_c)^2)^{3/2}} = |f''(x_c)|.$$

① The general equation of a <u>parabola</u> is $f(x) = ax^2 + bx + c$. Then $f'(x) = 2ax + b$ and $f''(x) = 2a$. So we set $a = 3/2$ and choose b and c freely in order to get curvature 3 at the extreme point.

Note that b and c only cause a translation because

$$ax^2 + bx + c = a\Big(x + \underbrace{\tfrac{b}{2a}}_{\text{translation left}}\Big)^2 \underbrace{- \frac{b^2}{4a} + c}_{\text{translation up}}.$$

So this makes sense.

② With the general cubic equation $f(x) = ax^3 + bx^2 + cx + d$, we get $f'(x) = 3ax^2 + 2bx + c$ and $f''(x) = 6ax + 2b$. The critical points are $x_c = \dfrac{-b \pm \sqrt{b^2 - 3ac}}{3a}$ (solve $f'(x) = 0$)

and $f''(x_c) = 2(-b \pm \sqrt{b^2-3ac}) + 2b = \pm 2\sqrt{b^2-3ac}$. This shows that every cubic function has the same curvature at both of its extrema. So the cubic in question does not exist. Here you could also argue that after an appropriate translation, we can assume that f has its point of inflection at $(0,0)$. This means that $f''(0) = 0$ and $f(0) = 0$, which imply $b = 0$ and $d = 0$. Then we are left with $f(x) = ax^3 + cx$ which is an odd function, i.e. symmetric w.r.t. reflection in the origin. So its two extrema have the same curvature.