p1

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1. If $r(t) = (x(t), y(t))$, then $r'(t) = (x'(t), y'(t))$

a) $r(t) = (\sin(t), \cos(3t))$, $t \in [0, \pi]$

$r'(t) = (\cos(t), -3\sin(3t))$

to sketch the curve, we find the endpoints
$r(0) = (\sin(0), \cos(0)) = (0, 1)$
$r(\pi) = (\sin(\pi), \cos(3\pi)) = (0, -1)$
and notice that if $t \in (0, \pi)$ then $\sin(t) > 0$.

now, we find that $\cos(3t) = 0 \Leftrightarrow t = \frac{m\pi}{3} + \frac{\pi}{6}$,
$m \in \mathbb{Z}$. only $m = 0, 1, 2$ give t values in the domain of $r(t)$.
$m = 0 \Rightarrow t = \frac{\pi}{6}$ gives $(\sin(\frac{\pi}{6}), 0) = (\frac{1}{2}, 0)$
$m = 1 \Rightarrow t = \frac{\pi}{2}$ gives $(\sin(\frac{\pi}{2}), 0) = (1, 0)$
$m = 2 \Rightarrow t = \frac{5\pi}{6}$ gives $(\sin(\frac{5\pi}{6}), 0) = (\frac{1}{2}, 0)$

also, we find that $\cos(3t) = 1 \Leftrightarrow t = \frac{m2\pi}{3}$, $m \in \mathbb{Z}$
and $\cos(3t) = -1 \Leftrightarrow t = \frac{(2m+1)\pi}{3}$, $m \in \mathbb{Z}$. the only solutions in the domain of $r(t)$ are
$\cos(3t) = 1 \Rightarrow t = \frac{2\pi}{3}$ gives $(\sin(\frac{2\pi}{3}), 1) = (\frac{\sqrt{3}}{2}, 1)$
$\cos(3t) = -1 \Rightarrow t = \frac{\pi}{6}$ gives $(\sin(\frac{\pi}{3}), -1) = (\frac{\sqrt{3}}{2}, -1)$

now using that
$3t \in (0, \frac{\pi}{2}) \Rightarrow \cos(3t) > 0$
$3t \in (\frac{\pi}{2}, \frac{3\pi}{2}) \Rightarrow \cos(3t) < 0$
$3t \in (\frac{3\pi}{2}, 2\pi) \Rightarrow \cos(3t) > 0$

we sketch $r(t)$

p2

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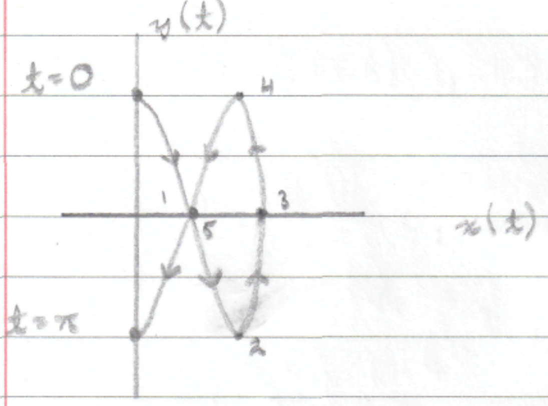

reflective symmetry in $x(t)$ axis.

$1 - t = \frac{\pi}{6}$, $4 - t = \frac{2\pi}{3}$

$2 - t = \frac{\pi}{3}$, $5 - t = \frac{5\pi}{6}$

$3 - t = \frac{\pi}{2}$

b) $r(t) = \left((1-t^2)\cos(4\pi t), (1-t^2)\sin(4\pi t), t\right)$, $t \in [-1, 1]$

$$r'(t) = \big(-4\pi(1-t^2)\sin(4\pi t) - 2t\cos(4\pi t),$$
$$4\pi(1-t^2)\cos(4\pi t) - 2t\sin(4\pi t),$$
$$1\big)$$

to sketch the curve, we consider the 2D projection $\bar{r}(t) = \left((1-t^2)\cos(4\pi t), (1-t^2)\sin(4\pi t)\right)$ of $r(t)$

notice that $(1-t^2) \geq 0$ on the domain of $r(t)$ and $1 - t^2 = 0 \Rightarrow t = \pm 1$

thus, both endpoints of $\bar{r}(t)$ are $(0,0)$

$\bar{r}(t)$ intercepts the x-axis when $\sin(4\pi t) = 0$

$\Rightarrow 4\pi t = m\pi$ where $m \in \mathbb{Z}$

$t = \frac{m}{4}$

thus we get new x intercepts when

$t = -3/4, -1/2, -1/4, 0, 1/4, 1/2, 3/4$

this gives the points (respectively)

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$$\left.\begin{array}{l}(-7/16,\ 0)\\(3/4,\ 0)\\(-15/16,\ 0)\end{array}\right\}\ t<0$$

$$(1,\ 0)\quad -\ t=0$$

$$\left.\begin{array}{l}(-15/16,\ 0)\\(3/4,\ 0)\\(-7/16,\ 0)\end{array}\right\}\ t>0$$

also, $\bar{z}(t)$ intercepts the y-axis when $\cos(4\pi t)=0$

$$\Rightarrow 4\pi t = m\pi + \frac{\pi}{2} \qquad \text{where } m\in\mathbb{Z}$$

$$t = \frac{m}{4} + \frac{1}{8}$$

thus we get new y intercepts when

$t = -7/8,\ -5/8,\ -3/8,\ -1/8,\ 1/8,\ 3/8,\ 5/8,\ 7/8$

this gives the points (respectively)

$$\left.\begin{array}{l}(0,\ 15/64)\\(0,\ -39/64)\\(0,\ 55/64)\\(0,\ -63/64)\end{array}\right\}\ t<0$$

$$\left.\begin{array}{l}(0,\ 63/64)\\(0,\ -55/64)\\(0,\ 39/64)\\(0,\ -15/64)\end{array}\right\}\ t>0$$

now, since $4\pi t$ increases as t increases, positive orientation is anticlockwise.

p3

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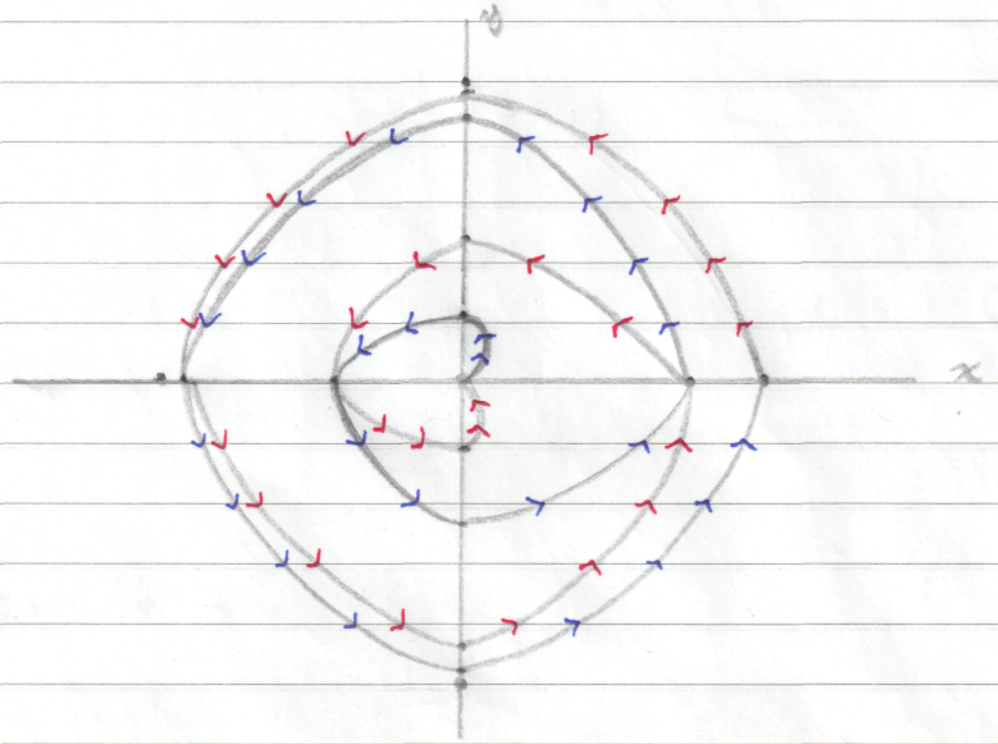
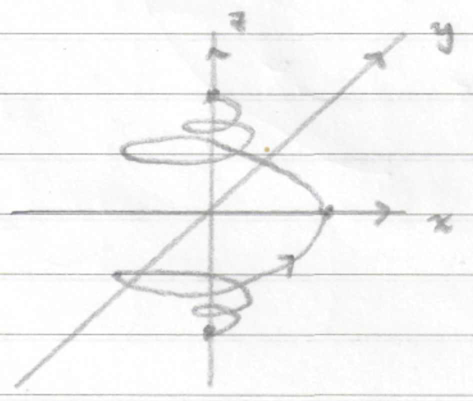
using blue for negative values of t and red for positive values of t, we find the graph of $\bar{R}(t)$ is approximately



this sketch of $\bar{R}(t)$ allows better visualisation of $R(t)$ which this resembles



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2. $P = (1/\sqrt{2}, 1/2, 1/2)$, $Q = (-1/\sqrt{2}, 1/2, -1/2)$

a) $l(t) = \left((1-t)/\sqrt{2} - t/\sqrt{2},\ \frac{(1-t)}{2} + t/2,\ (1-t)/2 - t/2 \right)$

$= \left((1-2t)/\sqrt{2},\ 1/2,\ (1-2t)/2 \right)$

$t \in [0,1]$ is a parametrisation of the line segment from P to Q.

b) the straight-line distance between 2 points $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ in $\mathbb{R}^3$ is given by

$|AB| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$

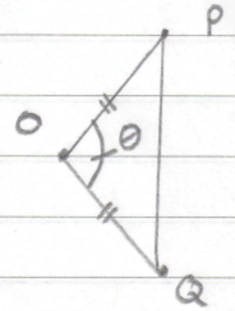
a point is on the unit sphere S if it is distance 1 from the origin $O = (0,0,0)$

$|OP| = \sqrt{1/2 + 1/4 + 1/4} = 1$

$|OQ| = \sqrt{1/2 + 1/4 + 1/4} = 1$

so P and Q are on the unit sphere.

the vectors P and Q define a plane. in this plane we have the triangle



to solve for θ, we first need the straight-line distance from P to Q

$|PQ| = \sqrt{(2/\sqrt{2})^2 + 1} = \sqrt{3}$

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now, we may solve for Θ using the cosine rule

$$(\sqrt{3})^2 = 1^2 + 1^2 - 2(1)(1)\cos\Theta$$
$$-1 = 2\cos\Theta$$
$$\cos\Theta = -1/2$$
$$\Rightarrow \Theta = 2\pi/3$$

the "equator" of the unit sphere S is parametrised by $(\cos(t), \sin(t), 0)$ where $t \in [0, 2\pi)$

we map the "equator" to the great circle through P and Q by finding a 3×3 matrix that maps $(\cos(0), \sin(0), 0) = (1,0,0)$ to P and maps $(\cos(\Theta), \sin(\Theta), 0) = (-1/2, \sqrt{3}/2, 0)$ to Q

now we have

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/2 \\ 1/2 \end{pmatrix}$$

by the rules of matrix multiplication,
$$a(1) = 1/\sqrt{2}$$
$$d(1) = 1/2$$
$$g(1) = 1/2$$

then $$\begin{pmatrix} 1/\sqrt{2} & b & c \\ 1/2 & e & f \\ 1/2 & h & i \end{pmatrix} \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1/2 \\ -1/2 \end{pmatrix}$$

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allows us to solve for b, e, h as

$$\left(\tfrac{1}{\sqrt{2}}\right)\left(-\tfrac{1}{2}\right) + b\left(\tfrac{\sqrt{3}}{2}\right) = -\tfrac{1}{\sqrt{2}}$$
$$b\left(\tfrac{\sqrt{3}}{2}\right) = \tfrac{1}{2\sqrt{2}} - \tfrac{1}{\sqrt{2}} = -\tfrac{1}{2\sqrt{2}}$$
$$b = -\tfrac{1}{2\sqrt{2}}\left(\tfrac{2}{\sqrt{3}}\right) = -\tfrac{1}{\sqrt{6}}$$

$$\left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right) + e\left(\tfrac{\sqrt{3}}{2}\right) = \tfrac{1}{2}$$
$$e\left(\tfrac{\sqrt{3}}{2}\right) = \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{3}{4}$$
$$e = \tfrac{3}{4}\left(\tfrac{2}{\sqrt{3}}\right) = \tfrac{3}{2\sqrt{3}} = \tfrac{\sqrt{3}}{2}$$

$$\left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right) + h\left(\tfrac{\sqrt{3}}{2}\right) = -\tfrac{1}{2}$$
$$h\left(\tfrac{\sqrt{3}}{2}\right) = -\tfrac{1}{2} + \tfrac{1}{4} = -\tfrac{1}{4}$$
$$h = -\tfrac{1}{4}\left(\tfrac{2}{\sqrt{3}}\right) = -\tfrac{1}{2\sqrt{3}}$$

now we map the equator to the great circle through P and Q by

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & c \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & f \\ \frac{1}{2} & -\frac{1}{2\sqrt{3}} & i \end{pmatrix} \begin{pmatrix} \cos(t) \\ \sin(t) \\ 0 \end{pmatrix}$$

notice that changing the values of c, f and i will have no effect

thus a parametrisation for the ~~great~~ great circle through P and Q is given by

$$\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{2}, \tfrac{1}{2}\right)\cos(t) + \left(-\tfrac{1}{\sqrt{6}}, \tfrac{\sqrt{3}}{2}, -\tfrac{1}{2\sqrt{3}}\right)\sin(t)$$
$$= \left(\frac{\cos(t)}{\sqrt{2}} - \frac{\sin(t)}{\sqrt{6}}, \frac{\cos(t) + \sqrt{3}\sin(t)}{2}, \frac{\cos(t)}{2} - \frac{\sin(t)}{2\sqrt{3}}\right)$$

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where $t \in [0, 2\pi)$. we cut the parametrisation to the parametrisation of the short arc from P to Q using the same parametrisation but with $t \in [0, 2\pi/3]$.

c) the length of the line segment from P to Q is the straight-line distance from P to Q, worked out previously.

$|PQ| = \sqrt{3}$

the length of the the short arc of the great circle from P to Q depends only on the angle $\Theta = 2\pi/3$ worked out earlier.

the arc length is $2\pi\left(\frac{2\pi/3}{2\pi}\right) = 2\pi/3$.

NOTE: b) can be solved by

$g(t) = \cos(t)P + \sin(t)Q_p$ where Q_p is found by

$w = Q - \left(P\cdot Q/P\cdot P\right)P$

and

$Q_p = w/\|w\|$.

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3. Let $r(t)$ be a parametric curve and $t = t(s)$ be a reparametrisation.

Suppose $r(t) = (x_1(t), \ldots, x_n(t))$. Now,

$dr/dt = (dx_1/dt, \ldots, dx_n/dt)$

If $t = t(s)$ is a reparametrisation,

$r(t(s)) = (x_1(t(s)), \ldots, x_n(t(s)))$ and

$$dr/ds = (dx_1/dt \; dt/ds, \ldots, dx_n/dt \; dt/ds)$$

$$= dt/ds \, (dx_1/dt, \ldots, dx_n/dt)$$

$$= dt/ds \; dr/dt \quad \text{by the chain rule.}$$

dt/ds is a scalar and so dr/ds is a scalar multiple of dr/dt. this implies that dr/ds and dr/dt are parallel vectors.

$\| dr/dt \| = ((dx_1/dt)^2 + \ldots + (dx_n/dt)^2)^{1/2}$ and

$$\| dr/ds \| = ((ds/dt)^2((dx_1/dt)^2 + \ldots + (dx_n/dt)^2))^{1/2}$$

$$= |ds/dt| \; \| dr/dt \|$$

thus the length of dr/ds is the length of dr/dt multiplied by the absolute value of ds/dt.

dr/dt is called velocity and $\| dr/dt \|$ is called speed.

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4. If $r : \mathbb{R} \to \mathbb{R}^3$ is a parametrisation, the curvature function $\kappa(t)$ of $r(t)$ is given by

$$\kappa(t) = \frac{\| r''(t) \times r'(t) \|}{\| r'(t) \|^3}$$

a) $r(t) = (t^2, \sin(t) - t\cos(t), \cos(t) + t\sin(t))$

$r'(t) = (2t, t\sin(t), t\cos(t))$

$r''(t) = (2, \sin(t) + t\cos(t), \cos(t) - t\sin(t))$

$$r''(t) \times r'(t) = \begin{vmatrix} i & j & k \\ 2 & \sin(t) + t\cos(t) & \cos(t) - t\sin(t) \\ 2t & t\sin(t) & t\cos(t) \end{vmatrix}$$

$$= i\left(t\cos(t)\left(\sin(t) + t\cos(t)\right) - t\sin(t)\left(\cos(t) - t\sin(t)\right)\right)$$

$$- j\left(2t\cos(t) - 2t\left(\cos(t) - t\sin(t)\right)\right)$$

$$+ k\left(2t\sin(t) - 2t\left(\sin(t) + t\cos(t)\right)\right)$$

$$= i\left(t\cos(t)\sin(t) + t^2\cos^2(t) - t\sin(t)\cos(t) + t^2\sin^2(t)\right)$$

$$- j\left(2t^2\sin(t)\right) + k\left(-2t^2\cos(t)\right)$$

$$= i\left(t^2\left(\cos^2(t) + \sin^2(t)\right)\right) + j\left(-2t^2\sin(t)\right) + k\left(-2t^2\cos(t)\right)$$

$$= \left(t^2, -2t^2\sin(t), -2t^2\cos(t)\right)$$

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$$\|r'(t)\| = \left(4t^2 + t^2\sin^2(t) + t^2\cos^2(t)\right)^{1/2}$$

$$= \left(4t^2 + t^2(\cos^2(t) + \sin^2(t))\right)^{1/2}$$

$$= (5t^2)^{1/2} = \sqrt{5}\,|t|$$

$$\|r''(t) \times r'(t)\| = \left(t^4 + 4t^4\sin^2(t) + 4t^4\cos^2(t)\right)^{1/2}$$

$$= \left(t^4 + 4t^4(\cos^2(t) + \sin^2(t))\right)^{1/2}$$

$$= (5t^4)^{1/2} = \sqrt{5}\,t^2$$

so

$$\kappa(t) = \frac{\sqrt{5}\,t^2}{(\sqrt{5}\,|t|)^3} = \frac{\sqrt{5}\,t^2}{5\sqrt{5}\,|t|^3} = \frac{1}{5|t|}$$

b) $r(t) = (t^2, 2t, \ln(t))$

$r'(t) = (2t, 2, 1/t)$

$r''(t) = (2, 0, -1/t^2)$

$$r''(t) \times r'(t) = \begin{vmatrix} i & j & k \\ 2 & 0 & -1/t^2 \\ 2t & 2 & 1/t \end{vmatrix}$$

$$= i(2/t^2) - j(2/t + 2t/t^2) + k(4)$$

$$= (2/t^2, -4/t, 4)$$

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$$\|r'(t)\| = \left(4t^2 + 4 + 1/t^2\right)^{1/2}$$

$$= \sqrt{\frac{4t^4+4t^2+1}{t^2}} = \frac{(4t^4+4t^2+1)^{1/2}}{|t|}$$

$$\|r''(t)\times r'(t)\| = \left(4/t^4 + 16/t^2 + 16\right)^{3/2}$$

$$= \sqrt{\frac{4(4t^4+4t^2+1)}{t^4}} = \frac{2(4t^4+4t^2+1)^{1/2}}{t^2}$$

so

$$\kappa(t) = \left(\frac{2(4t^4+4t^2+1)^{1/2}}{t^2}\right)\left(\frac{|t|^3}{(4t^4+4t^2+1)^{3/2}}\right)$$

$$= \frac{2|t|}{4t^4+4t^2+1}$$

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5. $r(t) = (2\sin(3t), t, 2\cos(3t))$

$r(t) = (0, \pi, -2)$ when $t = \pi$

the osculating plane of $r(t)$ is defined by the unit tangent vector

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

and the unit normal vector

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

$r'(t) = (6\cos(3t), 1, -6\sin(3t))$

$\|r'(t)\| = (1 + 36\cos^2(3t) + 36\sin^2(3t))^{1/2}$

$= \sqrt{37}$

so $T(t) = 1/\sqrt{37}\,(6\cos(3t), 1, -6\sin(3t))$

$T'(t) = 1/\sqrt{37}\,(-18\sin(3t), 0, -18\cos(3t))$

$\|T'(t)\| = \left(18^2/37\,\sin^2(3t) + 18^2/37\,\cos^2(3t)\right)^{1/2}$

$= 18/\sqrt{37}$

so $N(t) = \sqrt{37}/18\,(1/\sqrt{37})(-18\sin(3t), 0, -18\cos(3t))$

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$$= (-\sin(3t), 0, -\cos(3t))$$

a vector normal to this plane is given by the binormal vector

$$B(t) = T(t) \times N(t)$$

at $t = \pi$ we have

$$B(\pi) = T(\pi) \times N(\pi)$$

$$= 1/\sqrt{37}\,(6\cos(3\pi), 1, -6\sin(3t)) \times (-\sin(3\pi), 0, -\cos(3\pi))$$

$$= 1/\sqrt{37}\,(-6, 1, 0) \times (0, 0, 1)$$

$$= \begin{vmatrix} i & j & k \\ -6/\sqrt{37} & 1/\sqrt{37} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= i(1/\sqrt{37}) - j(-6/\sqrt{37}) + k(0)$$

$$= i/\sqrt{37} + 6j/\sqrt{37} + 0k$$

$$= (1/\sqrt{37}, 6/\sqrt{37}, 0)$$

since $B(\pi)$ is normal to the osculating plane, $\sqrt{37}\,B(\pi) = (1, 6, 0)$ is normal also.

now, the osculating plane is defined by
$$1(x-0) + 6(y-\pi) + 0(z-2) = 0$$

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this gives the plane $x + 6y = 6\pi$.

to find the osculating circle, we must find the curvature

$$\kappa(t) = \frac{\| r''(t) \times r'(t) \|}{\| r'(t) \|^3}$$

$$\| r'(t) \| = \sqrt{37}$$

$$r''(t) = (-18\sin(3t), 0, -18\cos(3t))$$

$$r''(t) \times r'(t) = \begin{vmatrix} i & j & k \\ -18\sin(3t) & 0 & -18\cos(3t) \\ 6\cos(3t) & 1 & -6\sin(3t) \end{vmatrix}$$

$$= i\,(18\cos(3t)) - j\,(108\sin^2(3t) + 108\cos^2(3t)) + k\,(-18\sin(3t))$$

$$= (18\cos(3t), -108, -18\sin(3t))$$

$$\| r''(t) \times r'(t) \| = \sqrt{18^2 + 108^2} = \sqrt{11988} = 18\sqrt{37}$$

so $$\kappa(t) = \frac{18\sqrt{37}}{37\sqrt{37}} = 18/37$$

and the radius of the osculating circle is

$$R(t) = 37/18$$

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the unit normal vector to $r(t)$ at $(0, \pi, -2)$ is $N(\pi) = (0,0,1)$ which points to the centre of the osculating circle

thus the centre of the osculating circle is
$(0, \pi, -2) + 37/18\,(0,0,1) = (0, \pi, 1/18)$

and the circle lies in the plane $x + 6y = 6\pi$

the osculating circle is given by

$C_\pi(\theta) = (0, \pi, 1/18) + \cos(\theta)\,T(\pi) + \sin(\theta)\,N(\pi)$

where $\theta \in [0, 2\pi)$