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1. $f(x,y) = x^2 + 3y^2$

solve $x^2 + 3y^2 = c$ for c in the range of f

range of f is $[0, \infty)$

$x^2 + 3y^2 = 0 \Rightarrow x^2 = y^2 = 0 \Rightarrow x = y = 0 \Rightarrow (x,y) = (0,0)$

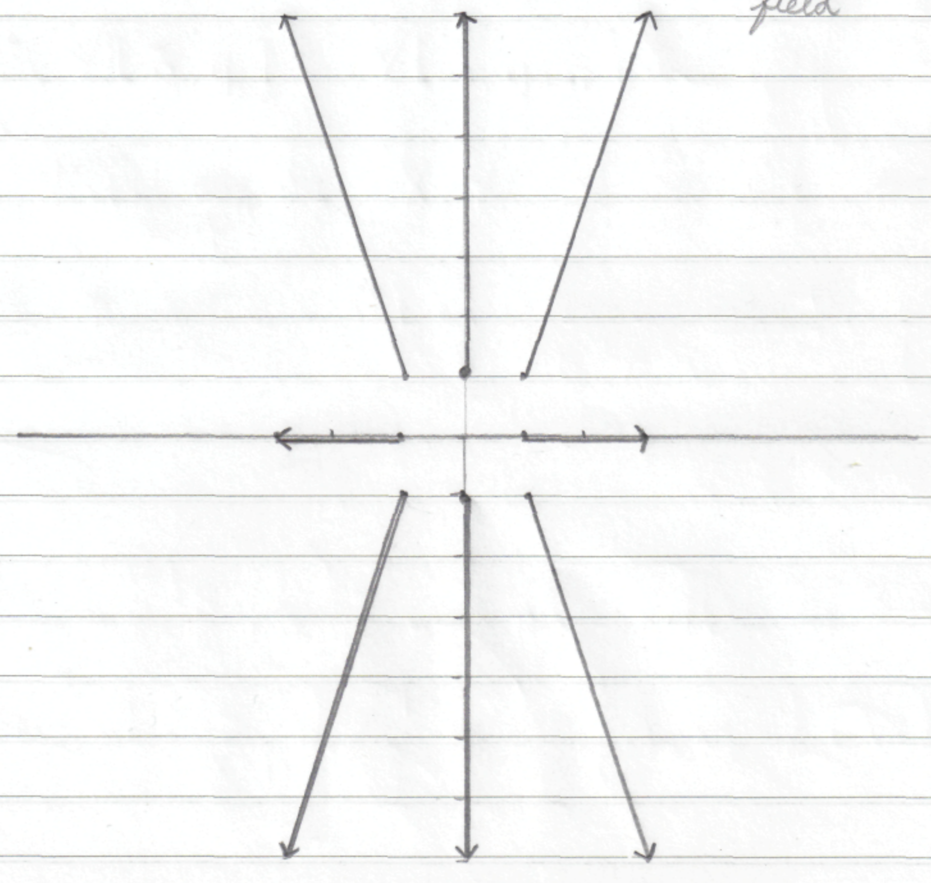
if c greater than 0

$x^2 + (\sqrt{3}y)^2 = c$ describes an ellipse

$\partial f/\partial x = 2x$
$\partial f/\partial y = 6y$

$\nabla f(x,y) = (2x, 6y)$

sketch of gradient field



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2a) $f(x,y) = (x-y)(xy^2 + x^2y)$

$= x^2y^2 - xy^3 + x^3y - x^2y^2$

$= x^3y - xy^3$

$\partial f/\partial x = 3x^2y - y^3$

$\partial f/\partial y = x^3 - 3xy^2$

b) $g(x,y,z) = 4z^2y - 5x^3y^2 + 2xyz$

$\partial g/\partial x = -15x^2y^2 + 2yz$

$\partial g/\partial y = 4z^2 - 10x^3y + 2xz$

$\partial g/\partial z = 8zy + 2xy$

c) $h(x,y) = \dfrac{\sin(x-2y)}{x+y} = (\sin(x-2y))(x+y)^{-1}$

$\partial h/\partial x = \cos(x-2y)(x+y)^{-1} + (\sin(x-2y))(-1)(x+y)^{-2}$

$= \dfrac{(x+y)\cos(x-2y) - \sin(x-2y)}{(x+y)^2}$

$\partial h/\partial y = (\cos(x-2y))(-2)(x+y)^{-1} + (\sin(x-2y))(-1)(x+y)^{-2}$

$= \dfrac{-2(x+y)\cos(x-2y) - \sin(x-2y)}{(x+y)^2}$

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3. Equation of tangent plane to $f(x,y)$ at (x_0, y_0)

$$\left(\partial f/\partial x\,(x_0,y_0)\right)(x-x_0) + \left(\partial f/\partial y\,(x_0,y_0)\right)(y-y_0) = z - z_0$$

where $z_0 = f(x_0, y_0)$

a) $f(x,y) = 3x^2y - xy^3 + e^{-x^2}$, point $(2, 1, f(2,1))$

$\partial f/\partial x = 6xy - y^3 + (e^{-x^2})(-2x)$
$\partial f/\partial y = 3x^2 - 3xy^2$

$f(2,1) = 3(2)^2(1) - 2(1)^3 + e^{-(2)^2} = 12 - 2 + e^{-4} = 10 + e^{-4}$

$\partial f/\partial x\,(2,1) = 6(2)(1) - (1)^3 - 2(2)e^{-(2)^2} = 11 - 4e^{-4}$

$\partial f/\partial y\,(2,1) = 3(2)^2 - 3(2)(1)^2 = 6$

$(11 - 4e^{-4})(x-2) + 6(y-1) = z - (10 + e^{-4})$

b) $g(x,y) = y\log(x+1) - x + 3y$, point $(0, 1, g(0,1))$

$\partial g/\partial x = y/(x+1) - 1$
$\partial g/\partial y = \log(x+1) + 3$

$g(0,1) = (1)\log(0+1) - 0 + 3(1) = 3$

$\partial g/\partial x\,(0,1) = 1/1 - 1 = 0$
$\partial g/\partial y = \log(1) + 3 = 3$

$0(x) + 3(y-1) = z - 3$

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4.a) $f(x,y) = x - x(y + \cos(x)) = x - xy - x\cos(x)$

$\partial f/\partial x = 1 - y - (\cos(x) - x\sin(x))$
$= 1 - y - \cos(x) + x\sin(x)$

$\partial f/\partial y = -x$

$df/du\,(x,y) = \nabla f(x,y) \cdot \frac{u}{\|u\|}$

$= \left(\partial f/\partial x\,(x,y)\right) u_1/\|u\| + \left(\partial f/\partial y\,(x,y)\right) u_2/\|u\|$

where $u = (u_1, u_2)$

if $u = (3,1)$, $\|u\| = \sqrt{3^2 + 1^2} = \sqrt{10}$

so
$df/du\,(x,y) = (1 - y - \cos(x) + x\sin(x)) \frac{3}{\sqrt{10}} + (-x)\frac{1}{\sqrt{10}}$

b) $df/dv\,(0,0) = \left(\partial f/\partial x\,(0,0)\right) v_1/\|v\| + \left(\partial f/\partial y\,(0,0)\right) v_2/\|v\|$

$= (1 - 0 - \cos(0) + 0\sin(0)) v_1/\|v\| + (-0)\, v_2/\|v\|$

$= \frac{0v_1 + 0v_2}{\|v\|} = 0$

which is true for all directions $v \neq (0,0)$

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5 $f(x,y) = \dfrac{4y^2+x}{x^2+y^2+1} = (4y^2+x)(x^2+y^2+1)^{-1}$

$$\partial f/\partial x = (4y^2+x)(-1)(x^2+y^2+1)^{-2}(2x) + 1\,(x^2+y^2+1)^{-1}$$

$$= \frac{(x^2+y^2+1) - 2x(4y^2+x)}{(x^2+y^2+1)^2}$$

$$\partial f/\partial y = (4y^2+x)(-1)(x^2+y^2+1)^{-2}(2y) + 8y\,(x^2+y^2+1)^{-1}$$

$$= \frac{8y(x^2+y^2+1) - 2y(4y^2+x)}{(x^2+y^2+1)^2}$$

notice $x^2+y^2+1 \neq 0$ for any $(x,y) \in \mathbb{R}^2$

so

$\partial f/\partial x\,(x,y) = 0 \iff (x^2+y^2+1) - 2x(4y^2+x) = 0$

and

$\partial f/\partial y\,(x,y) = 0 \iff 8y(x^2+y^2+1) - 2y(4y^2+x) = 0$

that is

$$x^2+y^2+1 - 8xy^2 - 2x^2 = 0 = y^2 - x^2 + 1 - 8xy^2$$

$$8yx^2 + \cancel{8y^3} + 8y - \cancel{8y^3} - 2yx = 0$$

$$2y(4x^2 - x + 4) = 0$$

$\Rightarrow$ $y = 0$ or $4x^2 - x + 4 = 0$

but $\sqrt{(-1)^2 - 4(4)(4)} \notin \mathbb{R}$ so $4x^2 - x + 4 \neq 0$

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thus $y = 0$ at a critical point

now $y^2 - x^2 + 1 - 8xy^2 = 0$

$= 0^2 - x^2 + 1 - 8x(0)^2 = 0 = 1 - x^2$

$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

so $f(x,y)$ has critical points at $(-1,0)$ and $(1,0)$

$\partial f/\partial x = (y^2 - x^2 + 1 - 8xy^2)(x^2+y^2+1)^{-2}$

$\partial f/\partial y = (8yx^2 + 8y - 2yx)(x^2+y^2+1)^{-2}$

$\partial^2 f/\partial x^2 = (-2x - 8y^2)(x^2+y^2+1)^{-2}$

$+ (y^2 - x^2 + 1 - 8xy^2)(-2)(x^2+y^2+1)^{-3}(2x)$

$\partial^2 f/\partial y^2 = (8x^2 - 2x + 8)(x^2+y^2+1)^{-2}$

$+ (8yx^2 + 8y - 2yx)(-2)(x^2+y^2+1)^{-3}(2y$

$\partial^2 f/\partial x^2 (1,0) = (-2)(1^2+1)^{-2}$

$+ (-(1)^2 + 1)(-2)(1^2+1)^{-3}(2(1))$

$= -2(2)^{-2} + 0(-2)(2)^{-3}(2)$

$= \frac{-2}{4} = -1/2 \Rightarrow (1,0)$ a max in x

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$\partial^2 f/\partial y^2 \,(1,0) = (8-2+8)(1+1)^{-2}$

$+ 0$

$= \frac{14}{4} = 7/2 \Rightarrow (1,0)$ a min in y

$\partial^2 f/\partial x^2 \,(-1,0) = 2(1+1)^{-2}$

$+ 0$

$= 2/4 = 1/2 \Rightarrow (-1,0)$ a min in x

$\partial^2 f/\partial y^2 \,(-1,0) = (8+2+8)(1+1)^{-2}$

$+ 0$

$= \frac{18}{4} = 9/2 \Rightarrow (-1,0)$ is a min in y

$\Rightarrow (1,0)$ is a saddle point
& $(-1,0)$ is a min.