The twisted cubic curve

The twisted cubic is defined by

$$r(t) = (t, t^2, t^3) \quad , \quad t \in \mathbb{R}.$$

We find

$$r'(t) = (1, 2t, 3t^2),$$

$$r''(t) = (0, 2, 6t) \quad \text{and}$$

$$|r'(t)| = \sqrt{1+4t^2+9t^4}.$$

So the unit tangent vector is

$$T(t) = \frac{1}{\sqrt{1+4t^2+9t^4}}(1, 2t, 3t^2).$$

Using $\kappa = \dfrac{|r'(t) \times r''(t)|}{|r'(t)|^3}$, we get

$$\kappa(t) = (1+4t^2+9t^4)^{-3/2} |(6t^2, -6t, 2)| = \frac{\sqrt{4+36t^2+36t^4}}{\sqrt{1+4t^2+9t^4}^{\,3}}.$$

Next, we calculate

$$T'(t) = \frac{(0,2,6t)\sqrt{1+4t^2+9t^4} - \frac{1}{2}(1+4t^2+9t^4)^{-1/2}(8t+36t^3)(1,2t,3t^2)}{1+4t^2+9t^4}$$

$$= \frac{(1+4t^2+9t^4)(0,2,6t) - (4t+9t^3)(1,2t,3t^2)}{(1+4t^2+9t^4)^{3/2}}$$

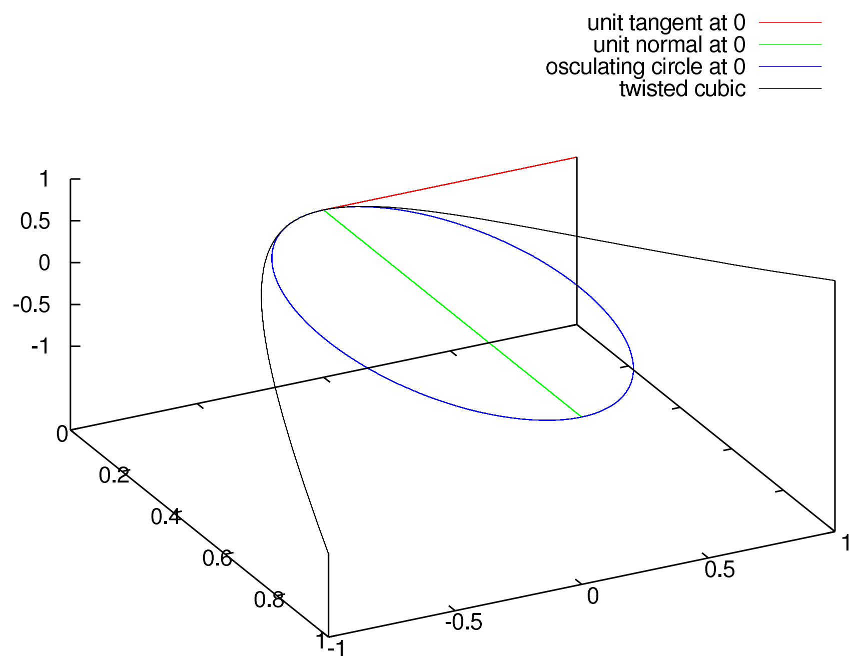
$$= \frac{1}{(1+4t^2+9t^4)^{3/2}}(-4t-9t^3, 2, 6t+12t^3+27t^5)$$

t	$r(t)$	$T(t)$	$N(t)$	$\kappa(t)$
0	$(0,0,0)$	$(1,0,0)$	$(0,1,0)$	2
$1/2$	$(1/2, 1/4, 1/8)$	$\frac{4}{\sqrt{41}}(1, 1, 3/4)$	$(-0.48, 0.31, 0.82)$	0.952
$-1/2$	$(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{8})$	$\frac{4}{\sqrt{41}}(1, -1, 3/4)$	$(0.48, 0.31, -0.82)$	0.952

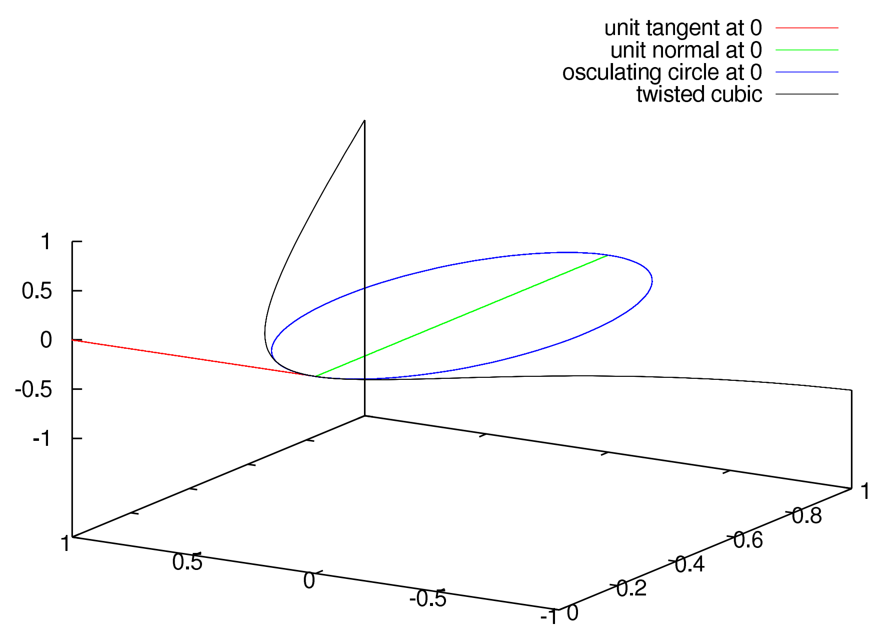
Pictures of Osculating Circles for the Twisted Cubic

The osculating circle at the origin

(i) viewed from above the xy-plane and

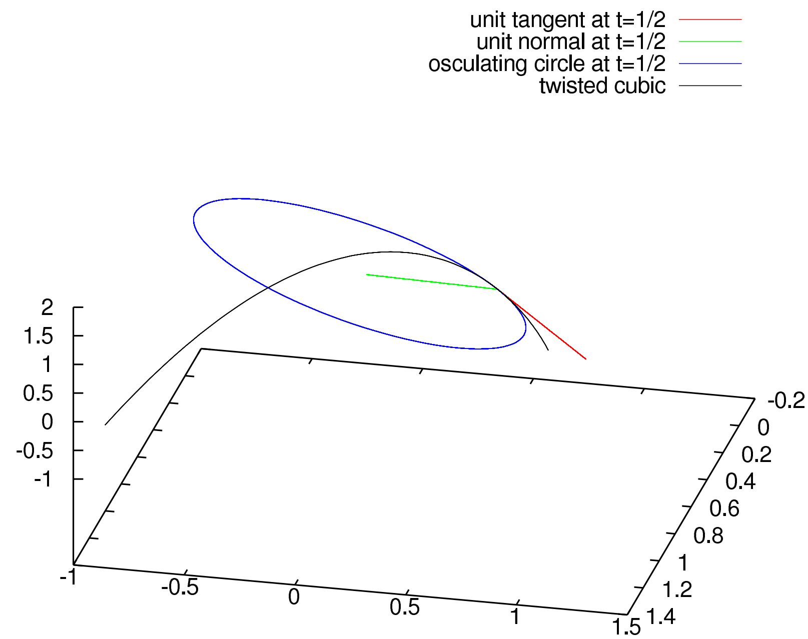


(ii) from a different angle.



The osculating circle at $t = 1/2$

(i) viewed from one side and



(ii) from the other side.