MA286 Analysis I – Course Summary

A real valued function of n variables is a function $f\colon D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^n$.

Examples. The air pressure at a point in $\mathbb{R}^3$, the height of terrain at a point in the plane, the volume of a rectangular box of height h, width w and depth d, etc.

The graph of f is $\{(x_1, \ldots, x_n, f(x_1, \ldots, x_n)) \mid (x_1, \ldots, x_n) \in D\} \subseteq \mathbb{R}^{n+1}$.

A level curve or level set of f is $\{(x_1, \ldots, x_n) \mid f(x_1, \ldots, x_n) = k\} \subseteq D$ with $k = const$.

Continuity. A function f as above is continuous at $(a_1, \ldots, a_n) \in D$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x_1, \ldots, x_n) - f(a_1, \ldots, a_n)| < \varepsilon$ for all $(x_1, \ldots, x_n)$ with $|(x_1, \ldots, x_n) - (a_1, \ldots, a_n)| < \delta$.

The partial derivative with respect to x_i of a function f as a above is defined by

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}$$

if this limit exists. It is caluclated like the derivative of a function of the single variale x_i, by treating all other variables as constants.

The tangent plane to the graph of f as above at $(a_1, \ldots, a_n)$ is given by

$$T(x_1, \ldots, x_n) = f(a_1, \ldots, a_n) + \sum_{i=1}^{n} (x_i - a_i) \frac{\partial f}{\partial x_i}(a_1, \ldots, a_n).$$

The gradient (or gradient field) of a function f as above is $\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$.

A vector field in $\mathbb{R}^n$ is function $F : \mathbb{R}^n \to \mathbb{R}^n$; it assigns to each point in $\mathbb{R}^n$ an n-dimensional vector. Note that $F = (f_1, \ldots, f_n)$ with $f_i\colon \mathbb{R}^n \to \mathbb{R}$.

Examples. Force fields, electromagnetic fields, the gradient field of a function $f\colon \mathbb{R}^n \to \mathbb{R}$.

The directional derivative in the direction of the unit vector $u = (u_1, \ldots, u_n)$ is the slope of the graph of f in the direction of u and is given as $D_u(f) = u \cdot \nabla f$, the dot product of u with the gradient vector of f.

The Hessian matrix of a function f as above is the matrix whose ij-entry is the second partial derivative $\frac{\partial^2 f}{\partial x_i \partial x_j}$.

Clairot's Theorem. If f has continuous second partial derivatives, then $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$.

A critical (or stationary) point of a function f as above is a point $(a_1, \ldots, a_n) \in D$ with $\nabla f(a_1, \ldots, a_n) = 0$.

Second Derivative Test in 2D. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function with critical point $a = (a_1, a_2)$. Let D be the determinant of the Hessian matrix of f evaluated at a and let f_{xx} be the second partial derivative of f with repect to x.

<u>If $D > 0$ and $f_{xx}(a) > 0$</u>, then f has a local minimum at a.

<u>If $D > 0$ and $f_{xx}(a) < 0$</u>, then f has a local maximum at a.

<u>If $D < 0$</u>, then f has a saddle point at a if its graph crosses the tangent plane at a.

A (parametric) curve in $\mathbb{R}^n$ is (the image of) a continuous function $r\colon [a,b] \longrightarrow \mathbb{R}^n$ where $a, b \in \mathbb{R}$ with $a < b$. Note that $r(t) = (x_1(t), \ldots, x_n(t))$.

The tangent vector to the curve $r(t)$ is its derivative $r'(t) = \dfrac{dr}{dt} = (x_1'(t), \ldots, x_n'(t))$.

A smooth curve is a curve $r(t)$ with continuous derivative.

The unit tangent vector is simply the normalised tangent vector $T(t) = \dfrac{r'(t)}{|r'(t)|}$.

The arc length of the curve $r(t)$ is $s(t) = \displaystyle\int_a^t |r'(p)|\, dp$. Hence $ds = |r'(t)|dt$.

The unit normal vector is the normalised derivative of the unit tangent vector $N(t) = \dfrac{T'(t)}{|T'(t)|}$.

The curvature of the curve $r(t)$ is the rate of change of the tangent vector with respect to arc length! It is given by $\kappa(t) = \dfrac{|T'(t)|}{|r'(t)|} = \dfrac{|r''(t) \times r'(t)|}{|r'(t)|^3}$.

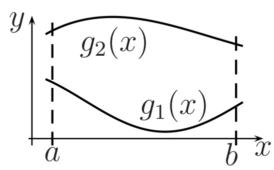
The osculating plane for the curve $r(t)$ at $r(t_0)$ is the plane through $r(t_0)$ spanned by the unit tangent and unit normal vectors at $r(t_0)$, i.e. $\{r(t_0) + \alpha T(t_0) + \beta N(t_0) \mid \alpha, \beta \in \mathbb{R}\}$.

The osculating circle for the curve $r(t)$ at $r(t_0)$ is the circle in the osculating plane that has radius $1/\kappa(t_0)$ and touches the curve at $r(t_0)$; it is the circle that best approximates the curve.

The line (or path) integral of a function $f\colon \mathbb{R}^n \to \mathbb{R}$ along the curve C parametrised by $r(t)$ for $a \le t \le b$ is $\displaystyle\int_C f\, ds = \int_a^b f(r(t))|r'(t)|\, dt$. If $f \ge 0$, the integral is the area of a wall erected on the curve and extending up to the graph of f.

The iterated integral of a function $f\colon \mathbb{R}^n \to \mathbb{R}$ over the rectangular region R where $a_i \le x_i \le b_i$ for $1 \le i \le n$ is $\displaystyle\int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} f\, dx_1 \cdots dx_n$, but the order can be changed.

The iterated integeral of $f\colon \mathbb{R}^2 \to \mathbb{R}$ over the region bounded by the lines $x = a$, $x = b$ and the graphs of $g_1(x)$ and $g_2(x)$ with $g_1 \le g_2$ is $\displaystyle\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$.



A simple closed curve is a curve which starts and ends at the same point and has no other self-intersections, i.e. $r(a) = r(b)$ and $r(t) \ne r(t')$ for $t \ne t'$.

Green's Theorem. Let $P, Q\colon \mathbb{R}^2 \to \mathbb{R}$ be functions whose partial derivatives exist. Let C be a positively oriented simple closed curve and let D be the region bounded by C. Then $\displaystyle\oint_C P\, dx + Q\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$. Special cases are $\displaystyle\iint_D dA = \oint_C x\, dy = -\oint_C y\, dx$.

The integral of a vector field $F\colon \mathbb{R}^n \to \mathbb{R}^n$ along the curve C, given as $r\colon [a,b] \to \mathbb{R}^n$, is $\displaystyle\int_C F \cdot dr$. If F is conservative with potential f, i.e. $F = \nabla f$, then $\displaystyle\int_C F \cdot dr = f(r(b)) - f(r(a))$, so that the integral, in this case, does not depend on the curve C but only on its end points.