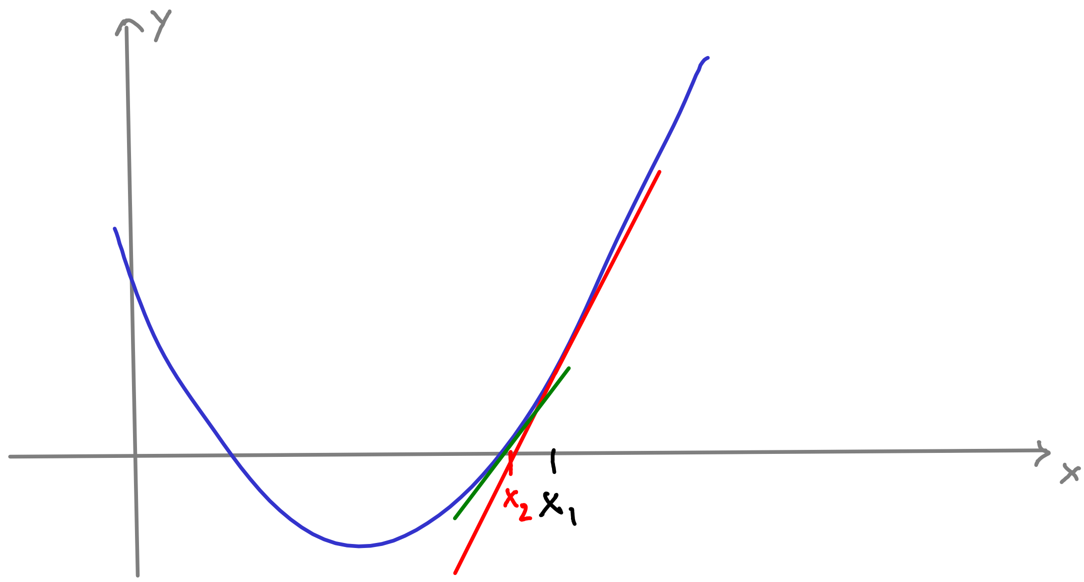
Calculus MA161/MA160 Sem II

Newton's Method

Consider a function $f(x)$ and the problem of finding a zero (or root) of it, that is a number a s.t. $f(a) = 0$.

For example $f(x) = x^2 - 2$. Can we approximate the solutions of $x^2 - 2 = 0$?



Idea: Find a first approximation x_1 (for example using the Intermediate Value Theorem)

① Then find the tangent to the graph of f at $(x_1, f(x_1))$.

② Use the x-axis intercept of this tangent as the next approximation.

Then repeat ① and ②

A bit of maths tells us that the equation for the tangent at x_1 is

$$t(x) = f'(x_1)\, x + c \quad \text{s.t.}$$

$$t(x_1) = f(x_1)$$

So $c = f(x_1) - f'(x_1)\, x_1$ and

$$t(x) = f'(x_1)(x - x_1) + f(x_1).$$

Hence $t(x) = 0$ implies

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Newton's formula is

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

, i.e. the next approximation x_{i+1} is given in terms of the last approximation x_i.

Example: $f(x) = x^2 - 2$. So $f'(x) = 2x$

Choose $x_1 = 2$ as first approximation. Then

$$x_2 = 2 - \frac{2}{4} = \frac{3}{2} = 1.5$$

$$x_3 = \frac{3}{2} - \frac{1}{12} = \frac{17}{12} = 1.4166$$

$$x_4 = 1.414215$$

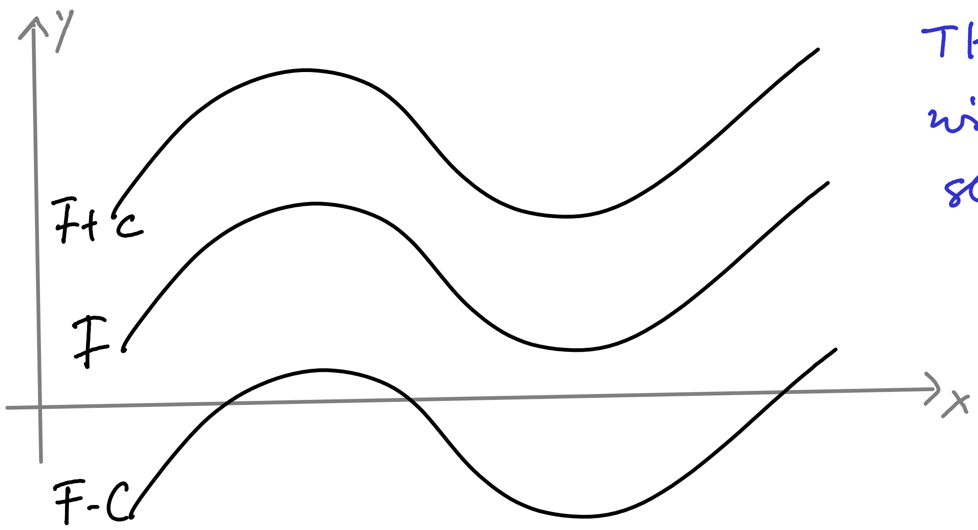
Antiderivatives

Definition: An antiderivative of a function f is a function F such that $F' = f$.

Examples:

$f(x)$	antiderivative of $f(x)$
x	$\frac{1}{2}x^2 + 5$
$\sin(x)$	$-\cos(x)$
e^x	e^x
$\ln(x)$	$x\ln(x) - x$ (not obvious)
$x^{-1} = \frac{1}{x}$	$\ln(x)$
x^m	$\frac{1}{m+1}x^{m+1}$ $\quad m \neq -1$

Theorem: Any two antiderivatives of f differ by a constant. In other words, if $F' = G' = f$, then $G - F = \text{const.}$



Three graphs with the same slope at every x.

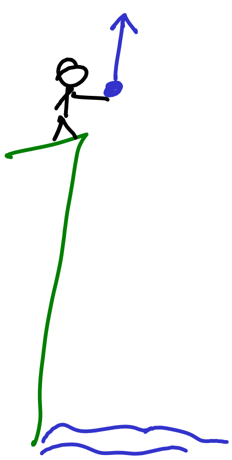
Proof: Suppose $G' = F' = f$. Then

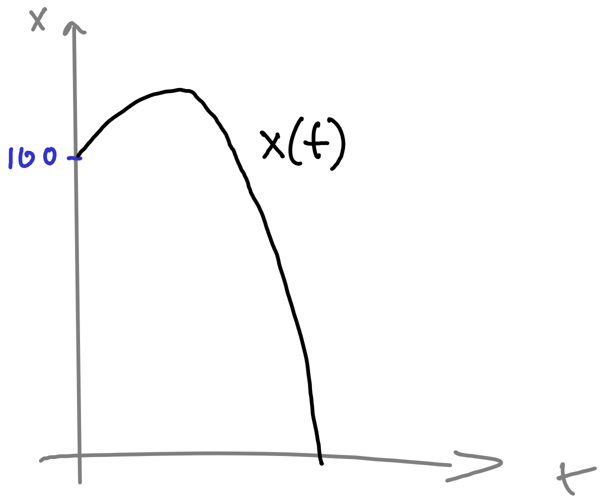
$$(G-F)' = G' - F' = 0,$$ so $G-F$ has a horizontal tangent everywhere and hence is itself a horizontal line. But that means $G - F = C = \text{const.}$

Antiderivatives come up in science, whenever it is easier to measure the derivative of the desired function. Examples are the velocity of a particle, whose antiderivative is the position.

Consequence: Having found an antiderivative F of f, then the most general antiderivative of f is $F + C$, where C is an arbitrary constant.

Example: Standing on the cliffs of Moher 100 m above sea level, you throw a stone vertically upwards with an initial speed of 3 m/s. When does the stone hit the water.





Gravity pulls the stone down : $a(t) = -9.8 \, m/s^2$

(a downwards accellerating force)

$v'(t) = a(t)$ gives

$$v(t) = -9.8 \, m/s^2 \; t + C$$

$$v(0) = C = 3 \, m/s \qquad \text{(our initial velocity)}$$

$$x'(t) = v(t) = 3 \, m/s - 9.8 \, m/s^2 \; t$$

$$x(t) = 3 \, m/s \; t - 4.9 \, m/s^2 \; t^2 + D$$

$$x(0) = D = 100 \, m \qquad \text{(initial position)}$$

So $x(t) = 100 + 3t - 4.9t^2$.

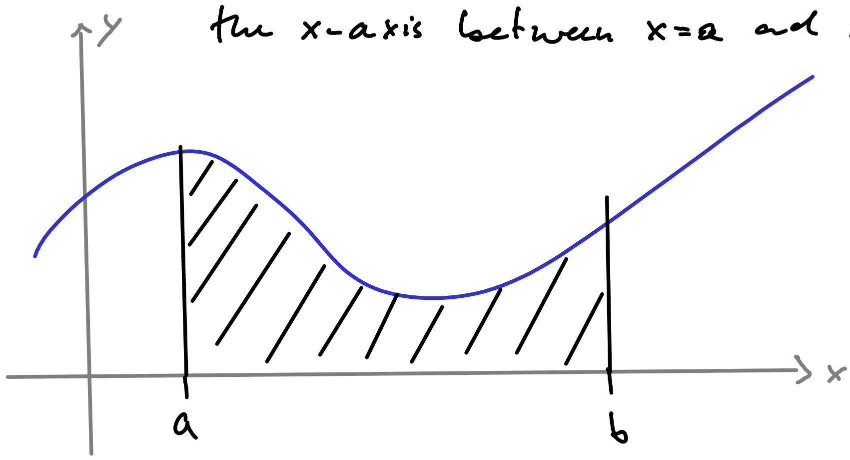
Now we need a zero of $x(t)$, i.e.

$$t^2 - \frac{3}{4.9} t - \frac{100}{4.9} = 0$$

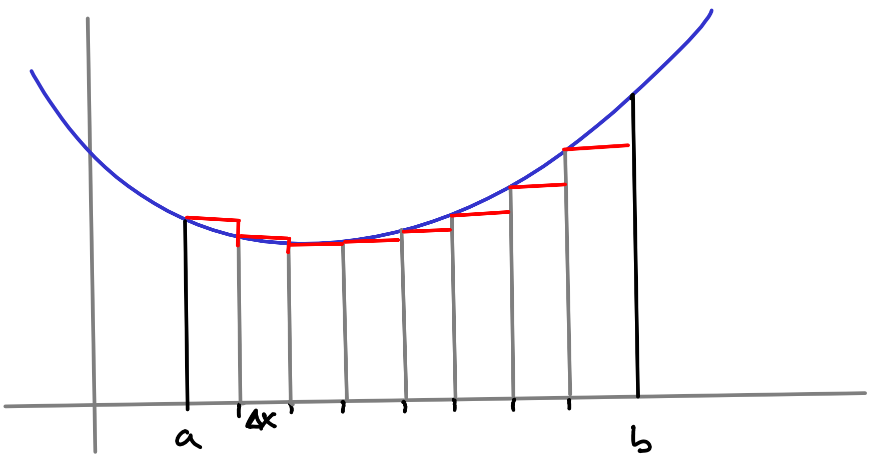
$$t = \frac{3}{9.8} \pm \sqrt{\frac{9}{(9.8)^2} + \frac{100}{4.9}} = 4.8 \qquad \text{(in seconds)}$$

The Area Problem

Suppose $f : \mathbb{R} \to \mathbb{R}$ with $f(x) \geq 0$ for all $x \in \mathbb{R}$
Fix $a, b \in \mathbb{R}$. What is the are below the graph of f and above the x-axis between $x = a$ and $x = b$?



The idea: Approximate the area by rectangles of a fixed width.



This approximation is obtained, using n rectangles each of with $\Delta x = \frac{b-a}{n}$ as

$$f(a)\Delta x + f(a+\Delta x)\Delta x + f(a+2\Delta x)\Delta x + \dots + f(a+(n-1)\Delta x)\Delta x$$

$$= \left[f(a) + f(a+\Delta x) + \dots + f(a+(n-1)\Delta x) \right] \Delta x$$

$$= \Delta x \sum_{k=0}^{n-1} f(a+k\Delta x)$$

__Definition:__ We define the (definite) __integral of $f:\mathbb{R}\to\mathbb{R}$ from a to b__ as the limit

$$\lim_{n\to\infty} \Delta x \sum_{k=0}^{n-1} f(a+k\Delta x), \quad \text{where } \Delta x = \frac{b-a}{a}.$$

The short notation for this limit, if it exists is

$$\int_a^b f(x)\,dx.$$

__Properties of integrals__ $(b>a)$

① If f is continuous, then $\int_a^b f(x)\,dx$ exists.

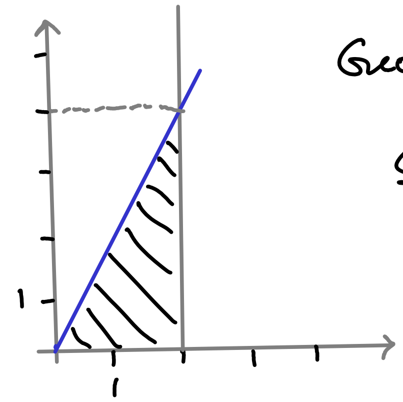
② $$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

③ If $f \geq 0$, then $\int_a^b f(x)\,dx \geq 0$

If $f \leq 0$, then $\int_a^b f(x)\,dx \leq 0$

④ $$\int_a^b C f(x)\,dx = C\int_a^b f(x)\,dx$$

Example: Let $f(x) = 2x$ and find $\int_0^2 f(x)\,dx$



Geometrically, the answer is 4.

Some approximations:

$\underline{n=2}$: $\Delta x = 1 = \frac{2-0}{2}$

$$1 f(0) + 1 f(1) = 0 + 2 = 2.$$

$\underline{n=3}$: $\Delta x = \frac{2}{3}$: $\frac{2}{3}\left(f(0) + f\left(\frac{2}{3}\right) + f\left(\frac{4}{3}\right)\right)$

$$= \frac{2}{3}\left(0 + \frac{4}{3} + \frac{8}{3}\right) = \frac{24}{9}$$

$n=n$: $\Delta x = \frac{2}{n}$, $\frac{2}{n}\sum_{k=0}^{n-1} f\left(k\frac{2}{n}\right) = \frac{2}{n}\sum_{k=0}^{n-1}\frac{4k}{n}$

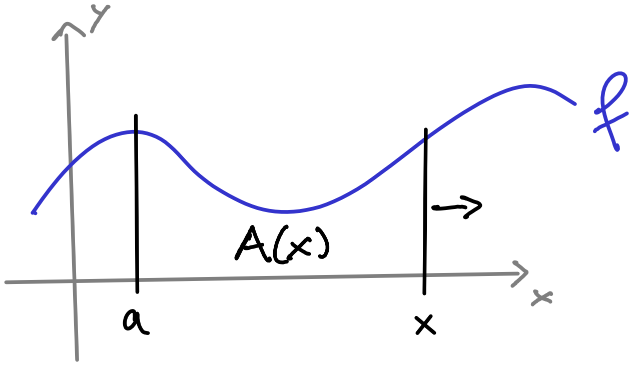
$$= \frac{8}{n^2}(0+1+2+\dots+n-1)$$

$$= \frac{8}{n^2}\,\frac{n(n-1)}{2}$$

$$= 4\,\frac{n-1}{n} \xrightarrow{n\to\infty} 4$$

Fundamental Theorem of Calculus

let's make the right hand boundary of an integral variable.



$$A(x) = \int_a^x f(t)\,dt$$

Intuitively, it seems that $\frac{dA}{dx}$, the rate of change of the area $A(x)$, depends on $f(x)$.

Fundamental Theorem of Calculus:

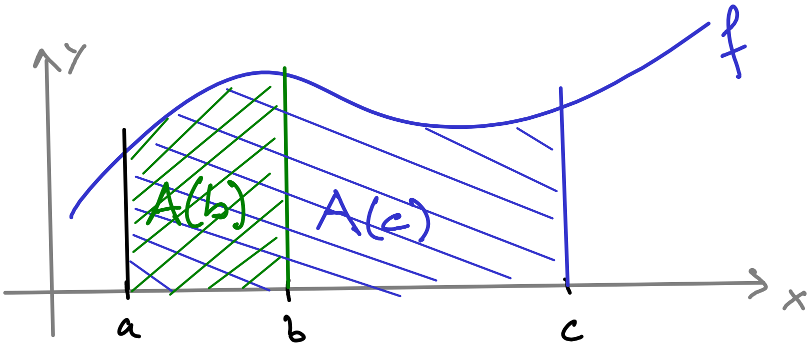
If $f(x)$ is continuous, and we define

$$A(x) = \int_a^x f(t)\,dt \quad , \quad \text{then}$$

$\frac{dA}{dx}(x) = f(x)$. In other words, $A(x)$ is an antiderivative of $f(x)$.

Consequences: If $B(x) = \int_b^x f(t)\,dt$, then $B(x)$ is an antiderivative of f, and A and B differ by the constant $\int_a^b f(t)\,dt$.

② $\int_b^c f(t)\,dt = A(c) - A(b)$



Applications

① Find the derivative of $F(x) = \int_a^x \sqrt{1+t^2}\,dt$.

By the Fundamental Theorem of Calculus, it is simply $\frac{dF}{dx}(x) = \sqrt{1+x^2}$

② Find the derivative of $F(x) = \int_a^{x^4} \sqrt{1+t^2}\,dt$

Make a substitution $z = x^4$, then look at

at $F(x) = F(z(x)) = \int_a^z \sqrt{1+t^2}\,dt$

Now $\frac{dF}{dx} = \frac{dF}{dz}\frac{dz}{dx}$ by the chain rule.

The FTC says that $\frac{dF}{dz} = \sqrt{1+z^2}$ and $\frac{dz}{dx} = 4x^3$,

so $\frac{dF}{dx}(x) = 4x^3\sqrt{1+x^8}$.

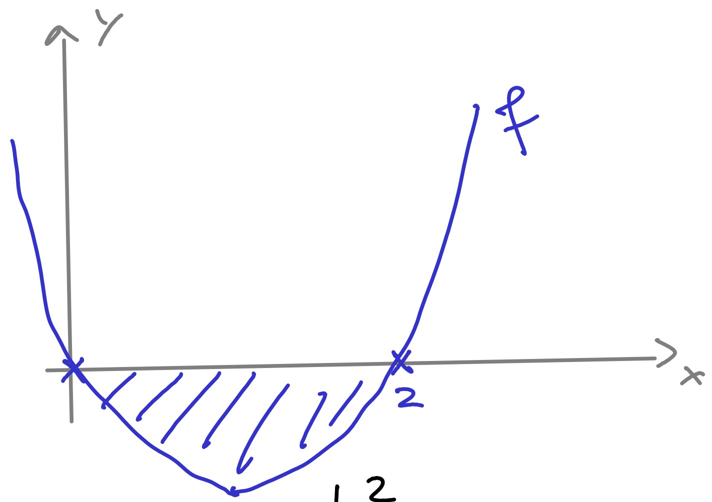
Upshot: Substitute x^4 for t and multiply by $\frac{d}{dx}(x^4) = 4x^3$.

One more: Find $\frac{dF}{dx}$ if $F(x) = \int_a^{3x} \sin(e^t)\, dt$.

Just like above:

$$\frac{dF}{dx}(x) = 3 \sin(e^{3x})$$

③ Find the area under the graph of $f(x) = x^2 - 2x$ that is below the x-axis.



So the area is $\left| \int_0^2 (x^2 - 2x)\, dx \right|$. We know that

$F(x) = \frac{1}{3}x^3 - x^2$ is an antiderivative of $x^2 - 2x$

and so

$$\int_0^2 (x^2-2x)\, dx = F(2) - F(0) = \frac{1}{3}2^3 - 4 - 0$$

$$= \frac{8}{3} - 4 = \frac{-4}{3}$$ and the area is $\frac{4}{3}$.

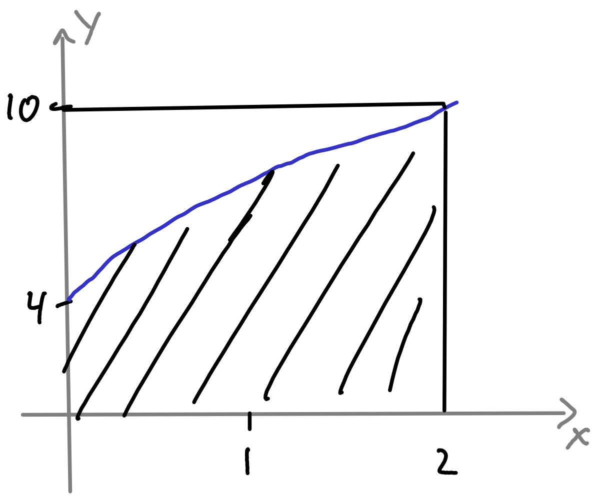
Some more examples of integrals

Calculate $\int_0^2 (x^3 - 2x^2 + 3x + 4)\,dx$ and interpret it as an area.

$$\int_0^2 (x^3 - 2x^2 + 3x + 4)\,dx = \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{3}{2}x^2 + 4x\right]_0^2$$

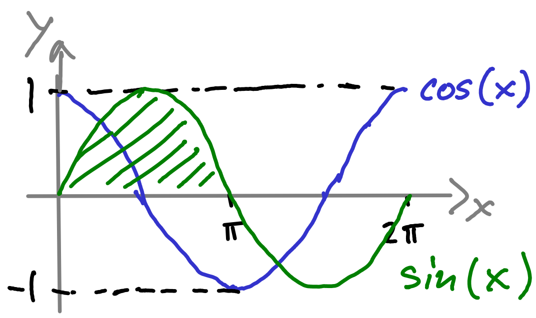
$$= 4 - \frac{16}{3} + 6 + 8 - (0) = 18 - \frac{16}{3} = \frac{38}{3}$$

It is the shaded area!



Evaluate $\int_0^{\pi} \sin(x)\,dx$.

Sol^n: $\int_0^{\pi} \sin(x)\,dx = \Big[-\cos(x)\Big]_0^{\pi} = 1 - (-1) = 2$



Obviously $\int_0^{2\pi} \sin(x)\,dx = 0$ (area below x-axis counts as negative area)

Integration by parts

Recall the product rule for derivatives:
Let f and g be functions, then

$$(fg)' = f'g + fg'.$$

Integrating both sides gives

$$\underbrace{\int (fg)'\,dx}_{= fg} = \int (f'g + fg')\,dx = \underline{\int f'g\,dx} + \int fg'\,dx$$

Consequently: $\int f'g\,dx = fg - \int fg'\,dx$

Example: Find $\int \underbrace{x}_{g}\,\underbrace{e^x}_{f'}\,dx$!

Sol^n: Let $g(x) = x$ and $f'(x) = e^x$. Then $g'(x) = 1$ and $f(x) = e^x$ and we get

$$\int xe^x dx = \int (g f')\,dx = fg - \int fg'\,dx = xe^x - \int e^x\,dx$$

$$= xe^x - e^x + C$$

Check: $\frac{d}{dx}(xe^x - e^x) = 1e^x + xe^x - e^x = xe^x$

Find $\int \ln(x)\,dx$ using integration by parts.

Sol^n: $\int \ln(x)\,dx = \int \underbrace{1}_{f'} \cdot \underbrace{\ln(x)}_{g}\,dx = x\ln(x) - \int x \frac{1}{x}\,dx$

$= x\ln(x) - \int 1\,dx = x\ln(x) - x + C$

Find $\int x\cos(x)\,dx$.

Sol^n: $\int \underbrace{x}_{g}\underbrace{\cos(x)}_{f'}\,dx = \underbrace{x}_{g}\underbrace{\sin(x)}_{f} - \int \underbrace{1}_{g'}\cdot\underbrace{\sin(x)}_{f}\,dx$

$= x\sin(x) + \cos(x) + C$

Find $\int x^3\cos(x)\,dx$.

Sol^n: $\int \underbrace{x^3}_{g}\underbrace{\cos(x)}_{f'}\,dx = x^3\sin(x) - \int \underbrace{3x^2}_{u}\underbrace{\sin(x)}_{v'}\,dx$

$= x^3\sin(x) - 3\left[-x^2\cos(x) - \int 2x(-\cos(x))\,dx\right]$

$= x^3\sin(x) + 3x^2\cos(x) - 6\int x\cos(x)\,dx$

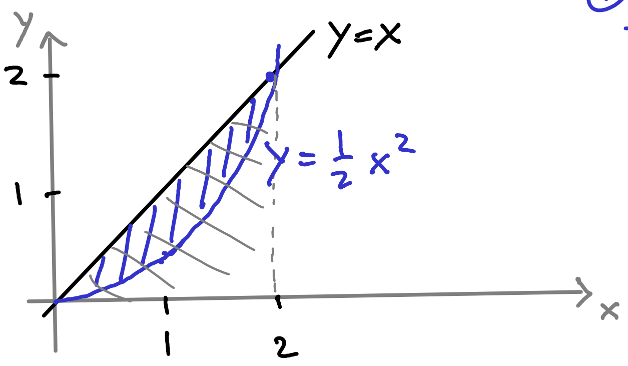
$= x^3\sin(x) + 3x^2\cos(x) - 6x\sin(x) - 6\cos(x) + C$

Check: $3x^2\sin(x) + x^3\cos(x) + 6x\cos(x) - 3x^2\sin(x)$

$- 6\sin(x) - 6x\cos(x) + 6\sin(x) = x^3\cos(x)$

Area between curves

Problem: Calculate the area between the graphs of $y = x$ and $y = \frac{1}{2}x^2$.



① Where do the graphs meet?

Set the two equations/expressions in x equal:

$$x = \frac{1}{2}x^2 \quad \text{or}$$

$$0 = x^2 - 2x = x(x-2)$$

So $x = 0$ or $x = 2$.

② Find out which curve is the upper bound.

Here it is $y = x$

③ The area is the integral of (upper − lower) from the left point of intersection to the right point of intersection.

$$\text{So} \int_0^2 \left(x - \frac{1}{2}x^2\right) dx = \left[\frac{1}{2}x^2 - \frac{1}{6}x^3\right]_0^2 = 2 - \frac{4}{3} - (0) = \frac{2}{3}.$$

Problem: Let $f(x) = x^2 - x$ and $g(x) = x + 1$ and find the area they enclose.

① Points of intersection: Solve $x^2 - x = x + 1$ or

$$\underline{x^2 - 2x - 1} = 0$$

$$x = 1 \pm \sqrt{2}$$

② Which function is the upper bound? Plug in a value between the points of intersection, say 1 in this case

$$f(1) = 0 < g(1) = 2$$

③ So the area is

$$\int_{1-\sqrt{2}}^{1+\sqrt{2}} x+1-(x^2-x)\,dx = \int_{1-\sqrt{2}}^{1+\sqrt{2}} (\underline{2x+1-x^2})\,dx$$

$$= \left[x^2 + x - \frac{1}{3}x^3 \right]_{1-\sqrt{2}}^{1+\sqrt{2}}$$

$$= 3+2\sqrt{2} + 1+\sqrt{2} \underline{- \frac{1}{3}(1+\sqrt{2})^3} - \left(3-2\sqrt{2}+1-\sqrt{2}-\frac{1}{3}(1-\sqrt{2})^3\right)$$

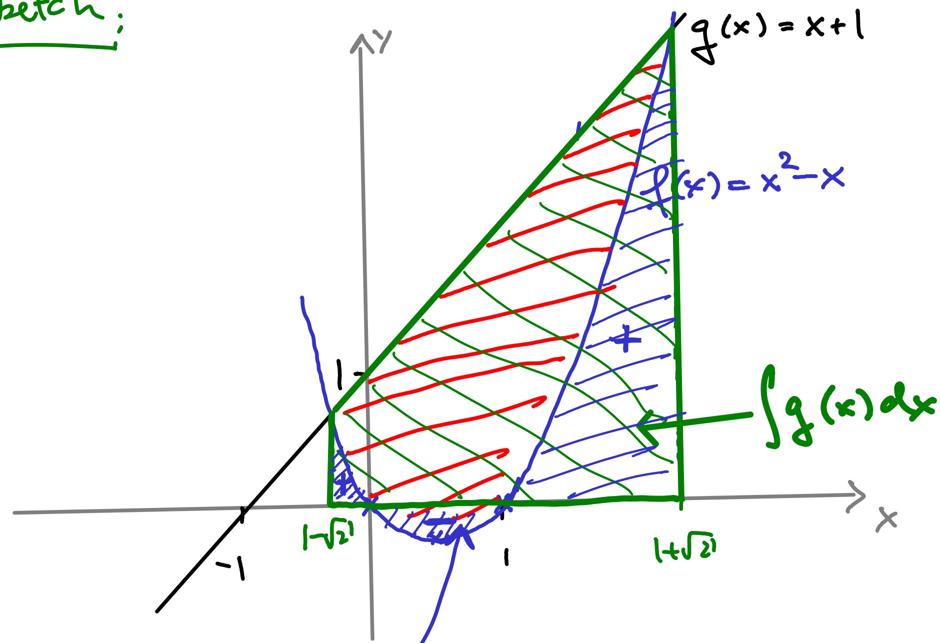
$$= 4\sqrt{2} + 2\sqrt{2} - \frac{1}{3}\left((1+\sqrt{2})^3 - (1-\sqrt{2})^3\right)$$

$$= 6\sqrt{2} - \frac{1}{3}\left(1+3\sqrt{2}+6+2\sqrt{2} - (1-3\sqrt{2}+6-2\sqrt{2})\right)$$

$$\left[\text{use } (a+b)^3 = a^3 + \underline{3}ab^2 + \underline{3}a^2b + b^3\right]$$

$$= 6\sqrt{2} - \frac{1}{3}(6\sqrt{2}+4\sqrt{2}) = \sqrt{2}\left(6-\frac{10}{3}\right) = \frac{8}{3}\sqrt{2}.$$

Sketch:



$f(x) = x^2 - x$
$\quad = x(x-1)$

$f(\frac{1}{2}) = -\frac{1}{4}$

Since area below the x-axis counts negatively, when we subtract $\int f(x)\,dx$ we actually add the area that's below the x-axis. So we get indeed the correct answer!

The Net Change Theorem

Theorem: The integral of a rate of change $F'(x)$ from a to b is the net change of $F(x)$, i.e.

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Examples: ① If water flows out of tank at a rate $\frac{dV}{dt}$ then $\int_{t_0}^{t_1} \frac{dV}{dt}\,dt = V(t_1) - V(t_0)$

② If $N(t)$ is the size of a population which grows at the rate $\frac{dN}{dt}$, then

$$\int_2^{20} \frac{dN}{dt}\,dt = N(20) - N(2)$$

③ Suppose a particle has velocity $v(t)$ at time t, then $\int_{t_0}^{t_1} v(t)\,dt$ = displacement of the particle from time t_0 to time t_1

If you want the total distance that particle covered between t_0 and t_1, then you need to calculate

$$\int_{t_0}^{t_1} |v(t)|\,dt$$

More specific examples:

① The rate of change of ~~a~~ charge of a capacitor (an electronic component that stores charge) is

$$\frac{dC}{dt}(t) = 1 - e^{-5t}$$

What is the net change of charge between $t_0 = 1$ and $t_1 = 5$?

Sol^n: The answer $C(5) - C(1)$ which is equal to

$$\int_1^5 \frac{dC}{dt}\, dt = \int_1^5 (1 - e^{-5t})\, dt$$

$$= \left[t + \frac{1}{5} e^{-5t} \right]_1^5$$

$$= 5 + \frac{1}{5} e^{-25} - \left(1 + \frac{1}{5} e^{-5}\right)$$

$$= 4 + \frac{1}{5}\left(e^{-25} - e^{-5}\right)$$

$$\frac{d}{dt}\left(e^{-5t}\right) = -5e^{-5t}$$

Why? Chain rule!

$u = -5t$

$g = e^u$

$$\frac{dg}{dt} = \frac{dg}{du}\,\frac{du}{dt}$$

$$= e^u \cdot (-5)$$

$$= e^{-5t}(-5) = -5e^{-5t}$$

Method of substitution

<u>Problem</u>: Find $\int x e^{x^2}\, dx$

Let's write $u = x^2$. Then $\frac{du}{dx} = 2x$ and we get

$du = 2x\, dx$, or $\frac{1}{2} du = x\, dx$.

Then $\int x e^{x^2} dx = \int e^{x^2} x\, dx = \int e^u \frac{1}{2} du$

$= \frac{1}{2}\int e^u\, du = \frac{1}{2} e^u + c = \frac{1}{2} e^{x^2} + c$, $c = \text{const.}$

<u>Check</u>: $\frac{d}{dx}\left(\frac{1}{2} e^{x^2} + c\right) = \frac{1}{2} e^{x^2} \frac{d}{dx}(x^2)$ Chain Rule

$= \frac{1}{2} e^{x^2} 2x = x e^{x^2}$

More substitutions

Problem: Find $\int t\sqrt{1+2t^2}\, dt$

Sol^n: Let $u = 1+2t^2$. Then $\frac{du}{dt} = 4t$, or

$\frac{1}{4}\, du = t\, dt$. Now we get

$$\int \sqrt{1+2t^2}\; t\, dt = \frac{1}{4}\int \sqrt{u}\, du = \frac{1}{4}\int u^{1/2}\, du$$

$$= \frac{1}{4}\,\frac{2}{3}\, u^{3/2} + c = \frac{1}{6}\left(1+2t^2\right)^{3/2} + c$$

Problem: Find $\int \frac{1}{1+x^2}\, dx$.

Sol^n: Let $x = \tan(u)$. Then

$$\frac{dx}{du} = \frac{d}{du}\left(\frac{\sin(u)}{\cos(u)}\right) = \frac{\cos^2(u) - (-\sin(u))\sin(u)}{\cos^2(u)} = \frac{1}{\cos^2(u)}$$

and so $dx = \frac{1}{\cos^2(u)}\, du$. Next

$$\int \frac{1}{1+x^2}\, dx = \int \frac{1}{1+\tan^2(u)}\,\frac{1}{\cos^2(u)}\, du$$

$$= \int \frac{1}{1+\frac{\sin^2(u)}{\cos^2(u)}}\cdot\frac{1}{\cos^2(u)}\, du$$

$$= \int \frac{1}{\cos^2(u)+\sin^2(u)}\, du = \int 1\, du = u + c$$

$$= \tan^{-1}(x) + c$$

Problem: Find $\int \sqrt{1-x^2}\,dx$ using the substitution $x = \sin(u)$.

Sol^n: $\frac{dx}{du} = \cos(u)$

$$\int \sqrt{1-x^2}\,dx$$

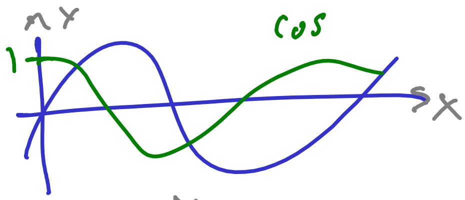
$$= \int \sqrt{1-\sin^2(u)}\,\cos(u)\,du$$

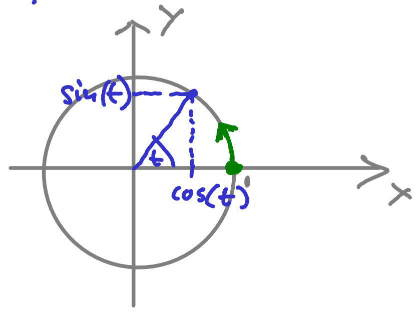
$$= \int \sqrt{\cos^2(u)}\,\cos(u)\,du$$

$$= \int \cos^2(u)\,du$$

$$= \frac{1}{2}\left(u + \frac{1}{2}\sin(2u)\right) + C$$

$$= \frac{1}{2}\left(\sin^{-1}(x) + \frac{1}{2}\sin\left(2\sin^{-1}(x)\right)\right) + C$$





$\sin^2(t) + \cos^2(t) = 1$

How did this happen? We used the tables.

Let's see if we can do this ourselves.

$$\int \cos^2(u)\,du$$

$$= \int \frac{1}{2}\left(\cos(2u)+1\right)du$$

$$= \int \frac{1}{2} + \frac{1}{2}\cos(2u)\,du$$

$$= \frac{1}{2}u + \frac{1}{4}\sin(2u)$$

another substitution $v = 2u$

Use the addition theorem for cosine.

$$\cos(2u) = \cos^2(u) - \sin^2(u)$$

$$= 2\cos^2(u) - \left(\cos^2(u) + \sin^2(u)\right)$$

$$= 2\cos^2(u) - 1$$

So $\cos^2(u) = \frac{1}{2}\left(\cos(2u)+1\right)$

Q4 Assignment 1

A particle travels along a straight line with velocity

$$v(t) = \underline{t^2 e^{-2t}} \quad \text{at time } t, \text{ measured in } \frac{m}{s}$$

How many meters has the particle travelled during the first t seconds?

<u>Sol^n</u>: This is the Net Change Theorem, since $v(t)$ is the rate of change of the position $p(t)$.

So we want $p(t) - p(0) = \displaystyle\int_0^t v(x)\,dx$

$$\boxed{\text{Integration by parts: } \int f'g\,dx = fg - \int fg'\,dx}$$

$$= \int_0^t \underbrace{x^2}_{f}\underbrace{e^{-2x}}_{g'}\,dx = \Big[\underbrace{x^2}_{f}\cdot\underbrace{\left(-\tfrac{1}{2}e^{-2x}\right)}_{g}\Big]_0^t - \int_0^t \underbrace{2x}_{f'}\underbrace{\left(-\tfrac{1}{2}e^{-2x}\right)}_{g}dx$$

$$= -\frac{1}{2}t^2e^{-2t} - 0 + \int_0^t \underbrace{x}_{u}\underbrace{e^{-2x}}_{v'}\,dx$$

$$\boxed{\frac{d}{dx}e^{-2x} = -2e^{-2x}}$$

$$= -\frac{1}{2}t^2e^{-2t} + \left[-\frac{1}{2}xe^{-2x}\right]_0^t - \int_0^t -\frac{1}{2}e^{-2x}\,dx$$

$$= -\frac{1}{2}t^2e^{-2t} - \frac{1}{2}te^{-2t} + \int_0^t \frac{1}{2}e^{-2x}\,dx$$

$$= -\frac{1}{2}e^{-2t}(t^2 + t) + \left[-\frac{1}{4}e^{-2x}\right]_0^t$$

Finding this mistake is tricky!

$$= -\frac{1}{2}e^{-2t}(t^2 + t) - \frac{1}{4}e^{-2t} + \frac{1}{4}$$

$$= -\frac{1}{2}e^{-2t}\left(t^2 + t + \frac{1}{2}\right) + \frac{1}{4}$$

Check: $\frac{d}{dx}\left(-\frac{1}{2}e^{-2t}\left(t^2+t+\frac{1}{2}\right)+\frac{1}{4}\right)$

$$= e^{-2t}\left(t^2+t+\frac{1}{2}\right) - \frac{1}{2}e^{-2t}(2t+1)$$

$$= e^{-2t}\left(t^2+t+\frac{1}{2} - t - \frac{1}{2}\right) = t^2e^{-2t} \checkmark$$

More integrals

Problem: Find $\displaystyle\int_0^2 x\sqrt{1+x}\,dx$.

Sol^n: Try a substitution: let $u = 1+x$ $= u(x)$. Then $du = dx$

and $x = u-1$. So

$3 = 1+2 = u(2)$
$1 = 1+0 = u(0)$

$$\int_0^2 x\sqrt{1+x}\,dx = \int_1^3 (u-1)\sqrt{u}\,du$$

$$= \int_1^3 \left(u^{3/2} - u^{1/2}\right) du = \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^3$$

$$= \frac{2}{5}\sqrt{3}^5 - \frac{2}{3}\sqrt{3}^3 - \left(\frac{2}{5} - \frac{2}{3}\right)$$

$$= \sqrt{3}\left(\frac{2}{5}\cdot 9 - \frac{2}{3}\cdot 3\right) + \frac{4}{15}$$

$$= \frac{1}{15}\left(24\sqrt{3} + 4\right)$$

Alternatively: Try integration by parts.

The formula $\int u\,dv = uv - \int v\,du$ (in the tables)

Preferably: $\displaystyle\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$

In our example:

$$\int_0^2 \underbrace{x}_{g}\underbrace{\sqrt{1+x}}_{f'}\,dx$$

$$\int\sqrt{1+x}\,dx = \frac{2}{3}(1+x)^{3/2}$$

$u = 1+x$ $\quad \| \qquad\qquad \|$

$$\int\sqrt{u}\,du = \frac{2}{3}u^{3/2}$$

$$= x\,\frac{2}{3}(1+x)^{3/2} - \int\frac{2}{3}(1+x)^{3/2}\,dx$$

$$= \left(\frac{2}{3}x(1+x)^{3/2} - \frac{4}{15}(1+x)^{5/2}\right]_0^2 = \frac{4}{3}\sqrt{3}^3 - \frac{4}{15}\sqrt{3}^5 + \frac{4}{15}$$

$$= \frac{1}{15}\left(\sqrt{3}\,(20\cdot 3 - 36) + 4\right) = \frac{1}{15}\left(24\sqrt{3} + 4\right)$$

Partial Fractions Method

Problem: Find $\displaystyle\int \frac{1}{x(1-x)}\,dx$

Sol^n: Idea: Write $\dfrac{1}{x(1-x)}$ as a sum $\dfrac{A}{x} + \dfrac{B}{1-x}$.

From $\dfrac{A}{x} + \dfrac{B}{1-x} = \dfrac{A(1-x) + Bx}{x(1-x)} = \dfrac{x(B-A) + A}{x(1-x)}$,

if we want this to be $\dfrac{1}{x(1-x)}$, then we need

$A = 1$ and $B - A = 0$. So $B = 1$ as well.

Hence $$\int \frac{1}{x(1-x)}\,dx = \int \left(\frac{1}{x} + \frac{1}{1-x}\right) dx$$

$$= \int \frac{1}{x}\,dx + \int \frac{1}{1-x}\,dx,$$

Use $u = 1-x$. Then $du = -dx$

$$= \ln|x| - \int \frac{1}{u}\,du$$

$$= \ln|x| - \ln|u| = \ln|x| - \ln|1-x|$$

$$= \ln\left|\frac{x}{1-x}\right|$$

Check: $$\frac{d}{dx}\left(\ln\left|\frac{x}{1-x}\right|\right) = \frac{1}{\frac{x}{1-x}} \cdot \frac{d}{dx}\left(\frac{x}{1-x}\right)$$ chain rule

$$= \frac{1-x}{x} \cdot \frac{1-x+x}{(1-x)^2} = \frac{1}{x(1-x)}$$ ✓

One more example: Find $\int \frac{2}{(x-1)(x-2)}\,dx$

Solve $\frac{A}{x-1} + \frac{B}{x-2} = \frac{A(x-2)+B(x-1)}{(x-1)(x-2)} = \frac{2}{(x-1)(x-2}$

for A and B.

$$= \frac{(A+B)x - 2A - B}{(x-1)(x-2)}$$

We need $A+B=0$ and $-2A-B=2$

$A=-B \implies -A=2$ and $B=+2$.

So $\int \frac{2}{(x-1)(x-2)}\,dx = \int \left(\frac{2}{x-2} - \frac{2}{x-1} \right) dx$

$$= 2\ln|x-2| - 2\ln|x-1| + c$$

Partial Fractions continued

Problem: Find $\displaystyle\int \frac{3\omega-1}{\omega+2}\,d\omega$

Observation: $\displaystyle\frac{d}{dx}\left(\ln(f(x))\right) = \frac{f'(x)}{f(x)}$

Examples:

- $\displaystyle\frac{d}{dx}\left(\ln(x+2)\right) = \frac{1}{x+2}$
- $\displaystyle\frac{d}{dx}\left(\ln(x^2+2x-3)\right) = \frac{2x+2}{x^2+2x-3}$

Solution: $\displaystyle\int \frac{3\omega-1}{\omega+2}\,d\omega = \int \frac{3(\omega+2)-7}{\omega+2}\,d\omega$

$$= \int \left(\frac{3(\omega+2)}{\omega+2} - 7\,\frac{1}{\omega+2}\right) d\omega$$

$$= \int 3\,d\omega - 7\int \frac{1}{\omega+2}\,d\omega$$

$$= 3\omega - 7\ln|\omega+2| + c$$

Check: $\displaystyle\frac{d}{d\omega}\left(3\omega - 7\ln|\omega+2|\right)$

$$= 3 - 7\,\frac{1}{\omega+2} = \frac{3\omega+6-7}{\omega+2} = \frac{3\omega-1}{\omega+2}$$

The above observation becomes

$$\ln(f(x)) = \int \frac{f'(x)}{f(x)}\,dx$$

when we integrate both sides.

Problem: Find $\displaystyle\int \frac{4x-4}{x^2-2x+5}\,dx$

Sol^n: $\displaystyle\int \frac{4x-4}{x^2-2x+5}\,dx = 2\int \frac{2x-2}{x^2-2x+5}\,dx$

$= 2\ln|x^2-2x+5| + c$

Problem: Find $\displaystyle\int \frac{x^3-1}{x^2+x}\,dx$

$$(x^3-1)/(x^2+x) = x-1+\frac{x-1}{x^2+x}$$

$$\begin{array}{r} x^3+x^2 \\ \hline -x^2-1 \\ -x^2-x \\ \hline x-1 \end{array}$$

$$3146/15 = 29+\frac{11}{15}$$

$$\begin{array}{r} 30 \\ \hline 146 \\ 135 \\ \hline 11 \end{array}$$

Check: $\left(x-1+\frac{x-1}{x^2+x}\right)(x^2+x) = (x-1)(x^2+x)+x-1$

$= x^3-x^2+x^2-x+x-1$

$= x^3-1$

Now go back to the problem:

$$\int \frac{x^3-1}{x^2+x}\,dx = \int \left(x-1+\frac{x-1}{x^2+x} \right) dx$$

$$= \frac{1}{2}x^2 - x + \int \frac{x-1}{x^2+x}\,dx$$

$$= \frac{1}{2}x^2 - x + \int \frac{1}{2}\ \frac{2x+1-3}{x^2+x}\,dx$$

$$= \frac{1}{2}x^2 - x + \frac{1}{2}\int \left(\frac{2x+1}{x^2+x} - \frac{3}{x^2+x} \right) dx$$

$$= \frac{1}{2}x^2 - x + \frac{1}{2}\ln|x^2+x| - \frac{1}{2}\int \frac{3}{x^2+x}\,dx$$

Aside: $\dfrac{3}{x^2+x} = \dfrac{3}{x(x+1)} = \dfrac{A}{x} + \dfrac{B}{x+1}$

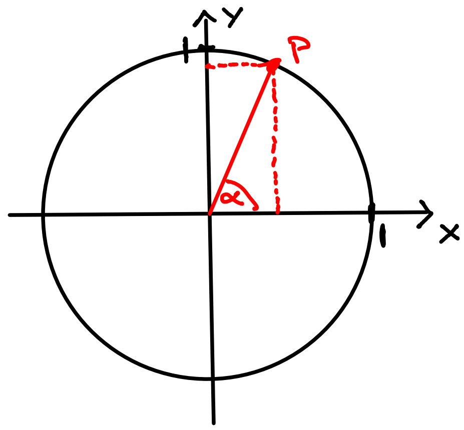
$$= \frac{(A+B)x + A}{x(x+1)}$$

$A = 3$ ad $A+B=0$, so $B=-3$

$$= \frac{1}{2}x^2 - x + \frac{1}{2}\ln|x^2+x| - \frac{1}{2}\int \left(\frac{3}{x} - \frac{3}{x+1} \right) dx$$

$$= \frac{1}{2}x^2 - x + \frac{1}{2}\ln|x^2+x| - \frac{3}{2}\ln|x| + \frac{3}{2}\ln|x+1| + c$$

Trigonometric Integrals

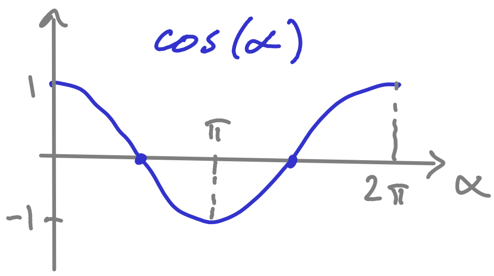


Unit circle.

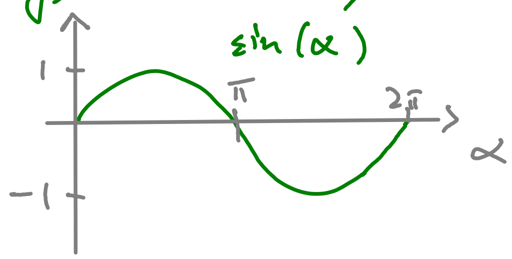
Measure angles in radians

$P = (\cos(\alpha), \sin(\alpha))$

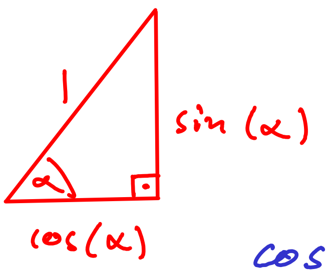
Cosine is the x-coordinate of a point on the unit circle.
When the point moves (with constant speed) anti-clockwise starting at $(1,0)$, i.e. $\alpha = 0$, then we get



Similarly, sine is the y-coordinate



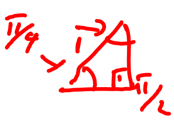
Identities:



$$\cos^2(\alpha) + \sin^2(\alpha) = 1$$

$\cos(\alpha) = 0$ at $\alpha = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$

i.e. $\alpha = \frac{2k+1}{2}\pi$ with $k \in \mathbb{Z}$



$\sin(\alpha) = 0$ at $\alpha = 0, \pm\pi, \pm 2\pi, \ldots$

i.e. $\alpha = k\pi$ with $k \in \mathbb{Z}$

Some common values:

α	0	$\pi/4$	$\pi/3$	$\pi/2$	π
$\cos(\alpha)$	1	$1/\sqrt{2}$	$1/2$	0	-1
$\sin(\alpha)$	0	$1/\sqrt{2}$	$\sqrt{3}/2$	1	0

$\cos(\alpha) = \cos(-\alpha)$, i.e. cosine is an *even* function

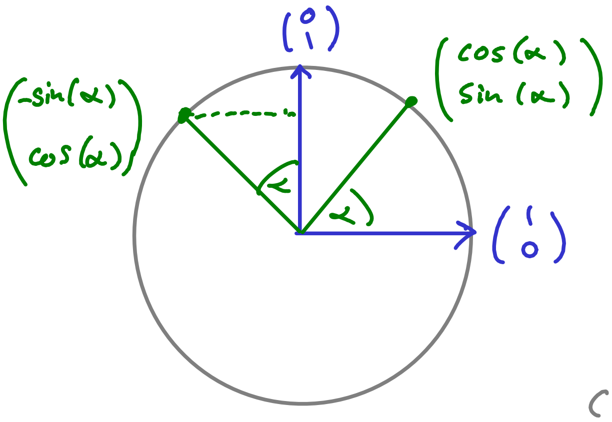
$\sin(\alpha) = -\sin(-\alpha)$, i.e. sine is an *odd* function.

$$\cos(\alpha) = \sin\left(\alpha + \tfrac{\pi}{2}\right)$$

In general: $A \sin(\omega t + \phi)$

A ← amplitute, ω ← frequency, ϕ ← phase

Rotations: A rotation about the origin by an angle α in the anti-clockwise direction can be represented by a 2×2 matrix



$$\rightsquigarrow R_\alpha = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

Check: $R_\alpha \binom{1}{0} = \binom{\cos(\alpha)}{\sin(\alpha)}$

$$R_\beta R_\alpha = R_{\alpha+\beta} = \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix} \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) & ? \\ \cos(\alpha)\sin(\beta) + \cos(\beta)\sin(\alpha) & ? \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}$$

So we find the Addition Theorems for sine & cosine:

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\sin(\alpha+\beta) = \cos(\alpha)\sin(\beta) + \cos(\beta)\sin(\alpha)$$

Setting $\alpha=\beta$ gives

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$$

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$$

The first one gives

$$\cos(2\alpha) = 2\cos^2(\alpha) - \left(\cos^2(\alpha) + \sin^2(\alpha)\right)$$

$$= 2\cos^2(\alpha) - 1$$

So $\cos^2(\alpha) = \frac{1}{2}\left(1+\cos(2\alpha)\right)$

Now find $\int \cos^3(x)\sin^4(x)\,dx$

Some trigonometric integrals

Problem: Find $\int \cos^3(x)\sin^4(x)\,dx$.

Observation: $\frac{d}{dx}\left(\sin^k(x)\right) = k\sin^{k-1}(x)\cos(x)$

Similarly $\frac{d}{dx}\left(\cos^k(x)\right) = -k\cos^{k-1}(x)\sin(x)$

$\cos^k(x) = (\cos(x))^k$
$k \geq 1$

Sol^n: $\int \cos^3(x)\sin^4(x)\,dx$

$= \int \cos^2(x)\cos(x)\sin^4(x)\,dx$

$= \int \left(1-\sin^2(x)\right)\cos(x)\sin^4(x)\,dx$ use $\cos^2(x) = 1-\sin^2(x)$

$= \int \left(\cos(x)\sin^4(x) - \cos(x)\sin^6(x)\right)dx$

$= \frac{1}{5}\sin^5(x) - \frac{1}{7}\sin^7(x) + c$

Problem: Find $\int \cos^8(x)\sin^5(x)\,dx$.

Sol^n: $\int \cos^8(x)\sin^5(x)\,dx$

$= \int \cos^8(x)\sin(x)\sin^4(x)\,dx$ use $\sin^2(x) = 1-\cos^2(x)$

$= \int \cos^8(x)\sin(x)\left(1-\cos^2(x)\right)^2 dx$

$= \int \cos^8(x)\sin(x)\left(1-2\cos^2(x)+\cos^4(x)\right)dx$

$= \int \left(\cos^8(x)\sin(x) - 2\cos^{10}(x)\sin(x) + \cos^{12}(x)\sin(x)\right)dx$

$= -\frac{1}{9}\cos^9(x) + \frac{2}{11}\cos^{11}(x) - \frac{1}{13}\cos^{13}(x) + c.$

Note: The last step is really a substitution:

$$\int \cos^{12}(x)\sin(x)\,dx \quad , \quad \text{use} \quad u = \cos(x)$$
$$du = -\sin(x)\,dx$$

$$= \int -u^{12}\,du = -\frac{1}{13}u^{13} + c = -\frac{1}{13}\cos^{13}(x) + c$$

Problem: Find $\int \cos^4(x)\sin^6(x)\,dx$

Sol^n: $\int \cos^4(x)\sin^6(x)\,dx$ use $\sin^2(x) = 1 - \cos^2(x)$

$$= \int \cos^4(x)\left(1 - \cos^2(x)\right)^3 dx$$

$$= \int \cos^4(x)\left(1 - 3\cos^2(x) + 3\cos^4(x) - \cos^6(x)\right)dx$$

$$= \int \left(\cos^4(x) - 3\cos^6(x) + 3\cos^8(x) - \cos^{10}(x)\right)dx$$

$$= \int \Bigg(\left(\tfrac{1}{2}(1+\cos(2x))\right)^2 - 3\left(\tfrac{1}{2}(1+\cos(2x))\right)^3 + 3\left(\tfrac{1}{2}(1+\cos(2x))\right)^4 - \left(\tfrac{1}{2}(1+\cos(2x))\right)^5\Bigg)dx$$

it get tideous!

Maybe there is a pattern?

$$\int \cos^2(x)\,dx = \int \tfrac{1}{2}(1+\cos(2x))\,dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin(2x) + c$$

$$\int \cos^4(x)\,dx = \int \left(\tfrac{1}{2}(1+\cos(2x))\right)^2 dx$$

$$= \int \tfrac{1}{4}\left(1 + 2\cos(2x) + \cos^2(2x)\right)dx$$

$$= \int \left[\frac{1}{4} + \frac{1}{2}\cos(2x) + \frac{1}{4}\,\frac{1}{2}\left(1+\cos(4x)\right)\right]dx$$

$$= \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{8}x + \frac{1}{16}\sin(4x) + c$$

Not so obvious what the pattern should be!

Strategy for $\int \sin^k(x)\cos^m(x)\,dx$

① If m is odd: Save one $\cos(x)$ and replace the others using $\cos^2(x) = 1 - \sin^2(x)$. Then use the substitution $u = \sin(x)$

② If k is odd: Save one $\sin(x)$ and replace the others using $\sin^2(x) = 1 - \cos^2(x)$. Then use the substitution $u = \cos(x)$.

③ If both k and m are even, then [replace $\sin^2(x)$ by $1 - \cos^2(x)$ or $\cos^2(x)$ by $1 - \sin^2(x)$ and] use half angle equalities:

$$\cos^2(x) = \frac{1}{2}\left(1 + \cos(2x)\right)$$

$$\sin^2(x) = \frac{1}{2}\left(1 - \cos(2x)\right)$$

Now try again!

$$\int \cos^4(x)\sin^6(x)\,dx$$

$$= \int \left[\tfrac{1}{2}(1+\cos(2x))\right]^2 \left[\tfrac{1}{2}(1-\cos(2x))\right]^3 dx$$

$$= \int \tfrac{1}{4}\left(1 + 2\cos(2x) + \cos^2(2x)\right)\tfrac{1}{8}\left(1 - 3\cos(2x) + 3\cos^2(2x) - \cos^3(2x)\right) dx$$

$$= \frac{1}{32}\int \left(1 - \cos(2x) - 2\cos^2(2x) + 2\cos^3(2x)\right) dx$$

$$= \frac{1}{32}\int \left(1 - \cos(2x) - (1+\cos(4x)) + 2\cos(2x)(1-\sin^2(2x))\right) dx$$

$$= \frac{1}{32}\left(x - \tfrac{1}{2}\sin(2x) - x - \tfrac{1}{4}\sin(4x) + \int \left[2\cos(2x) - 2\cos(2x)\sin^2(2x)\right] dx \right.$$

$$= \frac{1}{32}\left(-\tfrac{1}{2}\sin(2x) - \tfrac{1}{4}\sin(4x) + \sin(2x) - \tfrac{1}{3}\sin^3(2x)\right) + C$$

Check: The derivative is

$$\frac{1}{32}\left(-\cos(2x) - \cos(4x) + 2\cos(2x) - 2\cos(2x)\sin^2(x)\right)$$

Trigonometric substitution

Yesterday we found

$$\int \cos(\omega t)\, dt = \frac{1}{\omega}\sin(\omega t)$$

This is really a substitution: Put $u = \omega t$, then $du = \omega\, dt$ or $dt = \frac{1}{\omega} du$, so that

$$\int \cos(\omega t)\, dt = \int \cos(u)\frac{1}{\omega}\, du = \frac{1}{\omega}\sin(u) = \frac{1}{\omega}\sin(\omega t)$$

Similarly, $\int \sin(\omega t + \varphi)\, dt = -\frac{1}{\omega}\cos(\omega t + \varphi) + c$

small phi, capital Φ

using the substitution $u = \omega t + \varphi$.

Problem: Find $\int \sqrt{a^2 - x^2}\, dx$, $0 < a = \text{const}$.

Sol^n: Put $x = a\sin(\alpha)$. Then $\frac{dx}{d\alpha} = a\cos(\alpha)$ and

$$\int \sqrt{a^2-x^2}\, dx = \int \sqrt{a^2 - a^2\sin^2(\alpha)}\; a\cos(\alpha)\, d\alpha$$

$$= \int \sqrt{a^2(1-\sin^2(\alpha))}\; a\cos(\alpha)\, d\alpha$$

$$= \int a^2|\cos(\alpha)|\cos(\alpha)\, d\alpha$$

$$= \int a^2\cos^2(\alpha)\, d\alpha \qquad \text{if } \cos(\alpha) \geq 0$$

$$= a^2\int \frac{1}{2}(1+\cos(2\alpha))\, d\alpha$$

$$= \frac{a^2}{2}\left(\alpha + \frac{1}{2}\sin(2\alpha)\right) + c$$

$$= \frac{a^2}{2}\left(\arcsin\left(\frac{x}{a}\right) + \frac{1}{2}\sin\left(2\arcsin\left(\frac{x}{a}\right)\right)\right) + c$$

From: $x = a\sin(\alpha)$ we get $\frac{x}{a} = \sin(\alpha)$

$$\arcsin\left(\frac{x}{a}\right) = \alpha$$

arcsin is the inverse function of sin, i.e.

$$\arcsin(\sin(x)) = x = \sin(\arcsin(x))$$

Some people write $\sin^{-1}(x)$ for $\arcsin(x)$

not to be confused with $(\sin(x))^{-1} = \frac{1}{\sin(x)} \neq \arcsin(x)$

Inverse functions:

$$\begin{tikzcd} \mathbb{R} \arrow[r, "f"] & \mathbb{R} \arrow[r, "g"] & \mathbb{R} \\ x \arrow[r, mapsto] \arrow[rr, bend right, "g\circ f"'] & f(x) \arrow[r, mapsto] & g(f(x)) = (g\circ f)(x) \end{tikzcd}$$

Eg. $f(x) = x^2 + 3$ and $g(t) = \sin(t)$

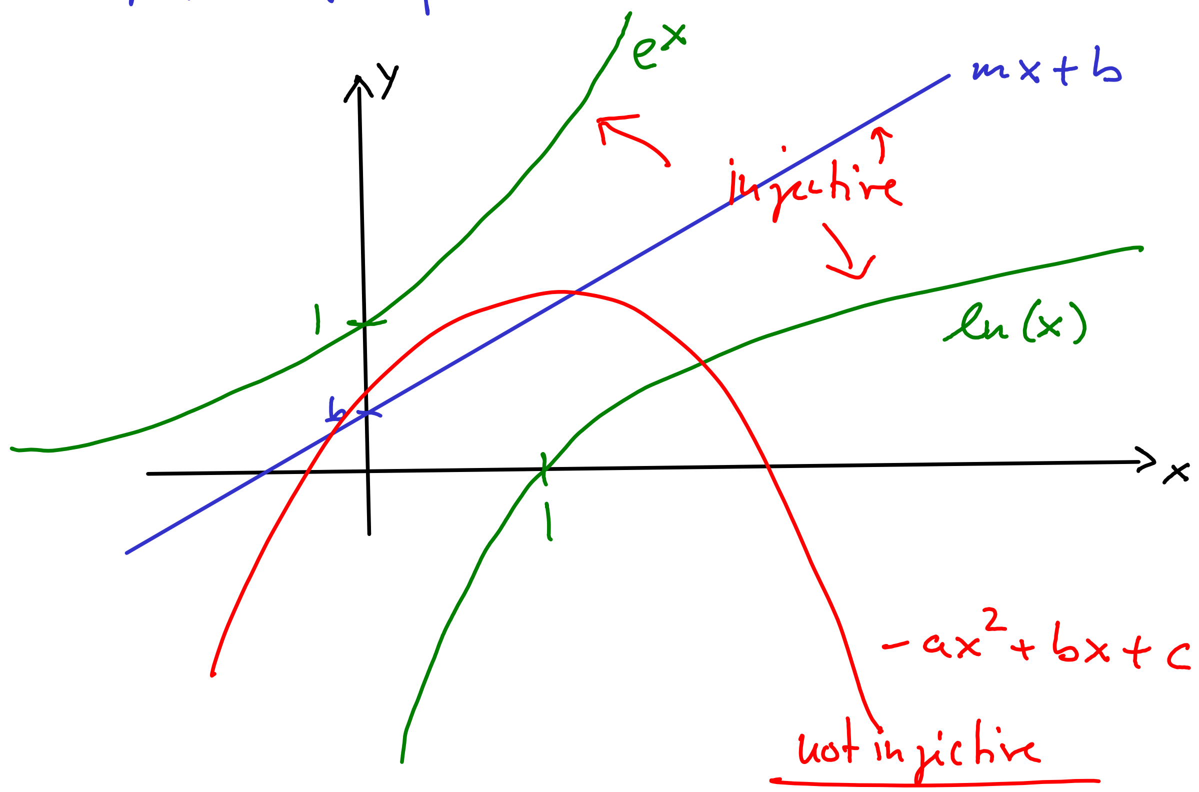
Then $(g\circ f)(x) = \sin(x^2 + 3)$

If $f(x_1) = f(x_2) = a$, then given only the information a, we cannot decide whether it was produced by f from x_1 or x_2.

Def: A function f is called <u>injective</u> if

$$f(x_1) = f(x_2) \text{ implies } x_1 = x_2 .$$

This means that every horizontal line meets the graph of f at most once.



Lines: $y = mx+b$, then $x = \dfrac{y-b}{m} = \dfrac{1}{m}y - \dfrac{b}{m}$

So if $f(x) = mx+b$, then $g(x) = \dfrac{1}{m}x - \dfrac{b}{m}$ is the inverse function of f.

$$(g \circ f)(x) = g(mx+b) = \frac{mx+b}{m} - \frac{b}{m} = x$$

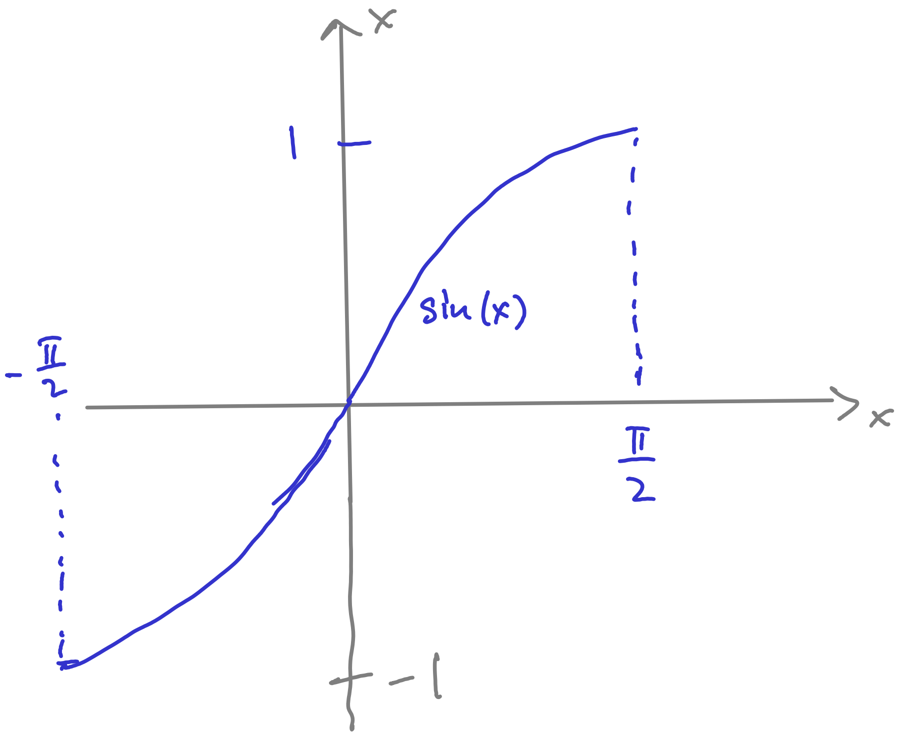
Logs & Exponentials:

$\ln(x)$ is the inverse function of $\exp(x) = e^x$

$$\ln(e^x) = x \quad \text{and} \quad e^{\ln x} = x$$

arcsin: $\arcsin(x)$ is only defined for $-1 \leq x \leq 1$

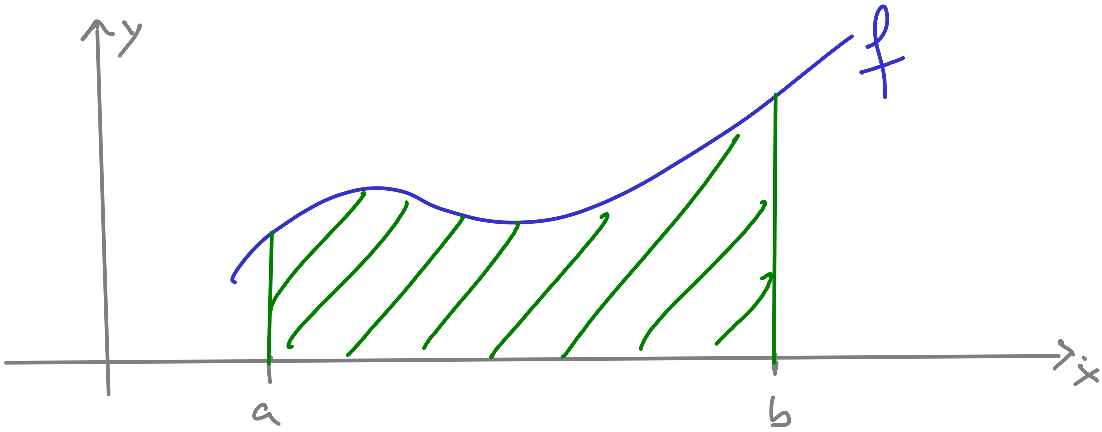
and produces only values $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

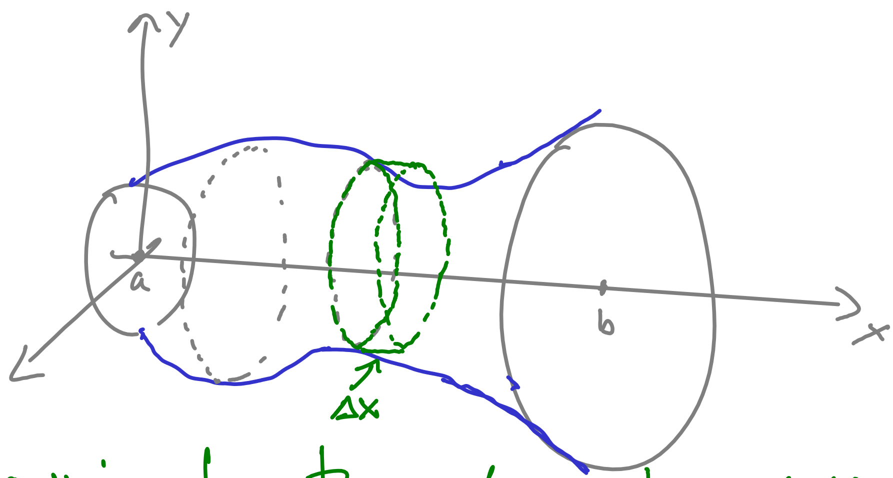


Volumes of revolution

Consider the following problem.

We want to know the volume of the solid obtained by rotating the area between the graph of a function f, the x-axis and the two lines $x=a$ and $x=b$ around the x-axis.





<u>Idea</u>: Approximate the volume by adding the volumes of thin discs.
Then take the limit when Δx, the thickness of the discs, shrinks to zero.

If we use n discs, then $\Delta x = \frac{b-a}{n}$, i.e.

$$a = x_0,\quad a + \Delta x = x_1,\ \ldots,\ x_i = a + i\Delta x,\ \ldots,\ b = x_n$$

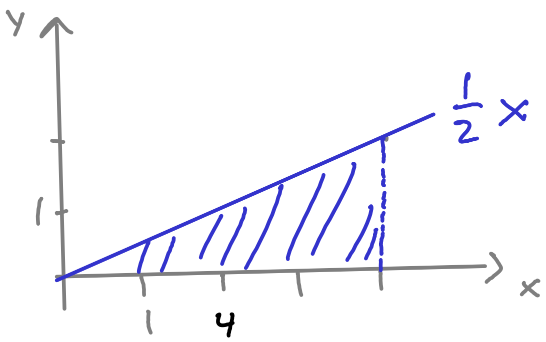
is our subdivision of the interval $[a,b]$, then the approximate volume is

$$V_n = \sum_{i=0}^{n-1} \pi \left[f(x_i)\right]^2 \Delta x .$$

When we let $n \to \infty$, then this becomes

$$V = \int_a^b \pi \left[f(x)\right]^2 dx .$$

Exaples: (i) A cone, say obtained by revolving the area underneath $f(x) = \frac{1}{2}x$ from 0 to 4.



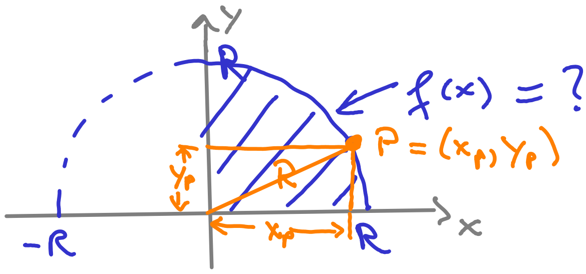
$$V = \pi \int_0^4 \left(\tfrac{1}{2}x\right)^2 dx = \pi \int_0^4 \tfrac{1}{4}x^2\, dx = \pi\left[\tfrac{1}{12}x^3\right]_0^4$$

$$= \frac{16}{3}\pi$$

Formula from tables:

$V_{cone} = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\, 2^2\, 4 = \frac{16}{3}\pi$

② Sphere (ball) of radius R.



$R^2 = x_p^2 + y_p^2$

Problem: Which function f should we take to get a quarter circle?

$$f(x) = y = \sqrt{R^2 - x^2}$$

So the volume of a ball of radius R is

$$V_{ball} = 2\pi \int_0^R \sqrt{R^2 - x^2}^{\,2} \, dx$$

$$= 2\pi \int_0^R (R^2 - x^2) \, dx$$

$$= 2\pi \left[R^2 x - \frac{1}{3} x^3 \right]_0^R$$

$$= 2\pi \left(R^3 - \frac{1}{3} R^3 - (0) \right)$$

$$= \frac{4}{3} \pi R^3$$

Homework Question 10 Assignment 2

Consider $\displaystyle\int \frac{2x}{x^2-1}\,dx$.

(1) Partial Fractions

$$\frac{2x}{x^2-1} = \frac{2x}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$= \frac{A(x+1) + B(x-1)}{x^2-1}$$

$$= \frac{x(A+B) + A - B}{x^2-1}$$

So $2 = A + B$ and $A - B = 0$.

$$\Downarrow$$

$$2 = 2A \Longleftarrow A = B$$

Hence $A = 1$ and $B = 1$.

$$\text{So} \int \frac{2x}{x^2-1}\,dx = \int \left(\frac{1}{x-1} + \frac{1}{x+1} \right) dx$$

$$= \ln|x-1| + \ln|x+1| + c$$

(2) Substitution: $w = x^2 - 1$. $dw = 2x\,dx$

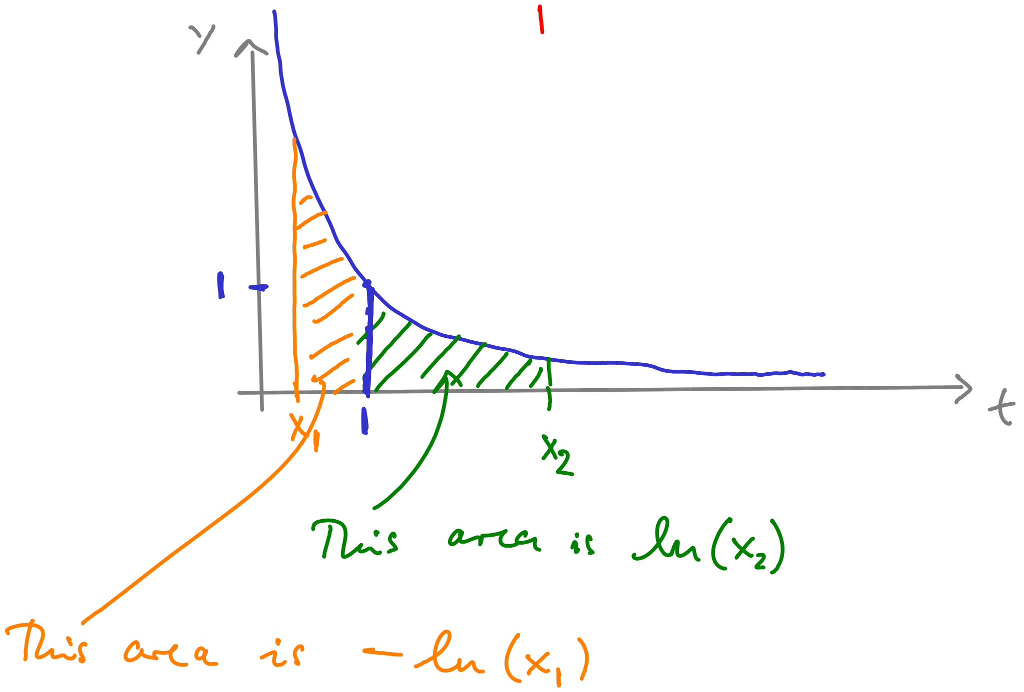
$$\text{So} \int \frac{2x}{x^2-1}\,dx = \int \frac{dw}{w} = \int \frac{1}{w}\,dw$$

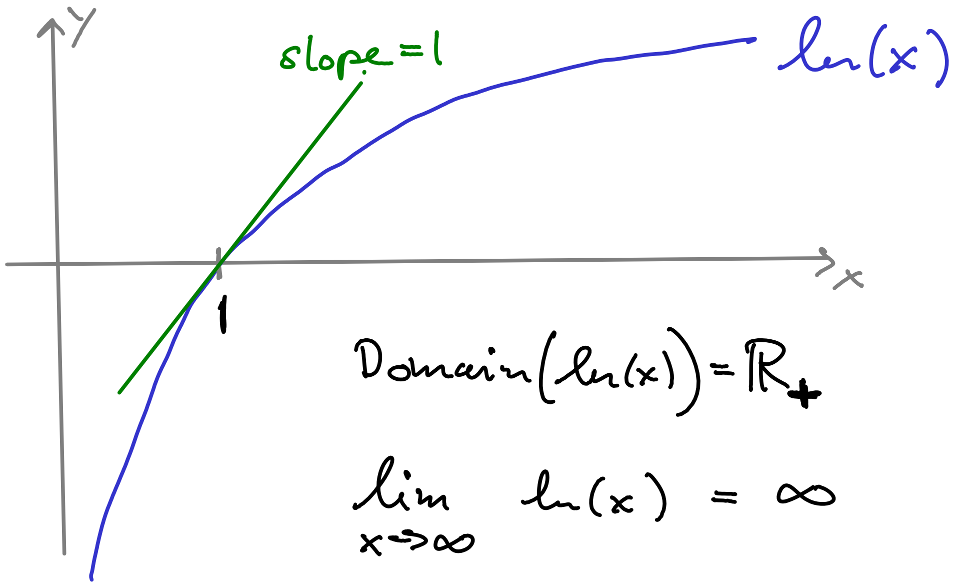
$$= \ln|w| + C = \ln|x^2-1| + c$$

Exponential and logarithmic Functions

The natural way to define the logarithm function is as

$$\ln(x) = \int_1^x \frac{1}{t}\,dt$$





$\text{Domain}(\ln(x)) = \mathbb{R}_+$

$$\lim_{x\to\infty} \ln(x) = \infty$$

$$\lim_{x\to 0^+} \ln(x) = -\infty$$

$\ln(x)$ is injective (one-to-one) and hence has an inverse function.

The inverse function of $\ln(x)$ is the exponential function. That means

$$y = \ln(x) \iff e^y = x$$

Also $$e^{\ln(x)} = x \quad \text{and} \quad \ln(e^y) = y$$

<u>Some facts:</u>

$$e^{a+b} = e^a e^b \qquad \ln(pq) = \ln(p) + \ln(q)$$

$$(e^a)^b = e^{ba} \qquad k\ln(p) = \ln(p^k)$$

<u>General exponentials</u>

Interest rate: If you get 3.75% interest every on an initial amount of E_0 euros, then after t years, you have

$$E(t) = E_0 \cdot (1.0375)^t \text{ euros.}$$

This is of the form $E(t) = K a^t$, a, K const.

Problem: What is $\frac{d}{dt}(ka^t)$?

Sol^n: $\frac{d}{dt}(ka^t) = K \frac{d}{dt}(a^t)$

$= K \frac{d}{dt}\left(e^{\ln(a)}\right)^t$ $\quad [a = e^{\ln(a)}]$

$= K \frac{d}{dt}\left(e^{\ln(a)t}\right)$ $\quad$ chain rule

$= K e^{\ln(a)t} \frac{d}{dt}(\ln(a)t)$

$= K e^{\ln(a)t} \ln(a)$

$= K \underline{\underline{\ln(a)}} a^t$

To sum up $\frac{d}{dt}(Ka^t) = K\ln(a)\, a^t$.

Differential Equations

A *differential equation* is an equation involving a function and some of its derivatives.

For example: ① $y' = xy$ where $y = y(x)$

② $y'' + 3y' + y = 0$

A solution of a differential equation is a function satisfying the equation. In general it is difficult to find a solution.

More realistic examples:

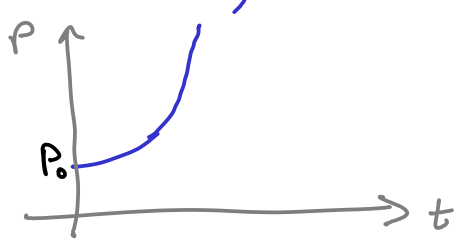
① *Population growth*: Since a species has more offspring the more individuals there are, we could expect the growth rate $\left(\frac{dP}{dt}\right)$ to be proportional to the actual population (P). This yields the differential equation $\frac{dP}{dt} = kP$, for some constant k.

Or $P' = kP$. $\quad k > 0$

TERMINOLOGY: P is the dependent variable

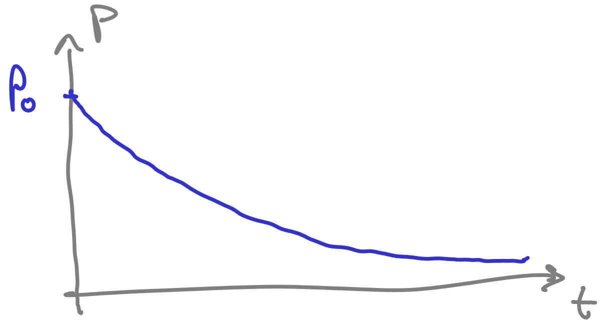
t is the independent variable

One solution is $P(t) = P_0 e^{kt}$, where $P_0 = P(0)$, the initial population.



Exponential Decay: Most prominently radioactive decay is governed by the equation $\frac{dP}{dt} = kP$, $k<0$.

Again, a solution is $P(t) = P_0 e^{kt}$



How does one come up with a solution??

A differential equation is called separable if it can be rewritten with all occurrences of the dependent variable one side and all occurrences of the independent variable one the other side.

For $\frac{dP}{dt} = kP$ this means that we rewrite it as

$\frac{1}{P} dP = k\,dt$ and now we can integrate

$\int \frac{1}{P} dP = \int k\,dt$ and we get

$$\ln|P| = \int k\,dt = kt + C, \quad C = \text{const.}$$

$$P = e^{\ln P} = e^{kt+C} = e^C e^{kt} = \hat{C} e^{kt}$$

Problem: A sample of radioactive material is measured twice, once at time $t=1$ and at time $t=5$ where it is found that $P(1) = 50$ and $P(5) = 48$.

① At what time is $P(t) = 35$?

② What is $P(20)$?

Sol^n: We know $P(t) = Ce^{kt}$ so

$$50 = P(1) = Ce^{k}$$
$$48 = P(5) = Ce^{5k}$$

$$\frac{48}{50} = \frac{Ce^{5k}}{Ce^{k}} = e^{4k} \Rightarrow 4k = \ln\left(\frac{48}{50}\right)$$
$$k = \frac{1}{4}\ln\left(\frac{48}{50}\right)$$
$$= -0.010205$$

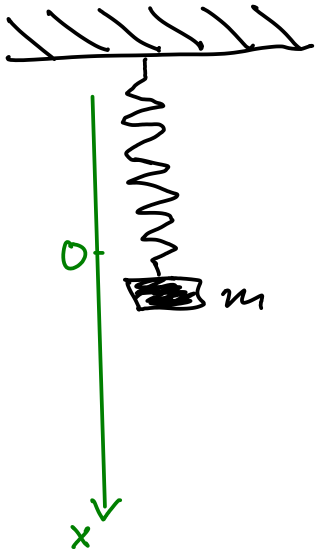
$$C = \frac{50}{e^{k}} = \frac{50}{\left(\frac{48}{50}\right)^{1/4}} = 50.5129$$

② $$P(20) = \frac{50}{\left(\frac{48}{50}\right)^{1/4}} e^{5\ln\left(\frac{48}{50}\right)} = 50\left(\frac{50}{48}\right)^{1/4}\left(\frac{48}{50}\right)^{5}$$

① Solve $35 = 50\left(\frac{50}{48}\right)^{1/4} e^{\frac{1}{4}\ln\left(\frac{48}{50}\right)t}$ for t:

$$\ln\left(\frac{35}{50}\left(\frac{48}{50}\right)^{1/4}\right) = \frac{1}{4}\ln\left(\frac{48}{50}\right)t.$$ So $t = 35.95$.

More differential equations



Express the force in two ways.

(1) Spring constant $k > 0$. The force generated by the spring is $-kx$

(2) Newton's law says

force = accelleration $\times$ mass, from which we get $m\dfrac{d^2x}{dt^2}$

So the motion is governed by $m\dfrac{d^2x}{dt^2} = -kx$, or

$$mx'' + kx = 0 \quad \text{or} \quad x'' = -\frac{k}{m}x.$$

One solution is $x(t) = A\sin\left(\sqrt{\frac{k}{m}}\,t\right)$ and another solution is $x(t) = B\cos\left(\sqrt{\frac{k}{m}}\,t\right)$ and so is

$$x(t) = A\sin\left(\sqrt{\tfrac{k}{m}}\,t\right) + B\cos\left(\sqrt{\tfrac{k}{m}}\,t\right)$$

It's a fact that this is the general solution to the differential equation. It is a superposition of two independent solutions which is all a 2^{nd} order diff. equ. can have.

The constants A and B are determined by initial conditions, like initial position and initial velocity, i.e.

$x(0) = B$ and $x'(0) = A\sqrt{\frac{k}{m}}$

More population growth

The simple minded $\frac{dP}{dt} = kP$ is not realistic.

No population will grow arbitrarily large.

A more realistic model assumes a capacity which is an upper bound and measures the proportion of that capacity. For small populations the growth should be exponential. One equation that captures these features is

$$\frac{dP}{dt} = kP(1-P)$$

which is called the <u>logistic equation</u>.

This is a separable equation:

$$\frac{1}{P(1-P)}\,dP = k\,dt$$ and we can integrate

$$\int \frac{1}{P(1-P)}\,dP = \int k\,dt = kt + C$$

$$\|$$

$$\int \left(\frac{1}{P} + \frac{1}{1-P}\right) dP = \ln P \cancel{+} \ln(1-P)$$

$$= \ln\left(P(1-P)\right)$$

WRONG!
Why?

This gives $P(1-P) = Ae^{kt}$, where $A = e^{C}$.

The logistic equation solved correctly:

The logistic equation is

$$\frac{dP}{dt} = kP(1-P)$$

We solve it by separating variables

$$\int \frac{1}{P(1-P)}\,dP = \int k\,dt = kt + c$$

$$\parallel$$

$$\int \left(\frac{1}{P} + \frac{1}{1-P}\right) dP = \ln(P) - \ln(1-P)$$

$$= \ln\left(\frac{P}{1-P}\right)$$

So $kt + c = \ln \frac{P}{1-P}$ and we get

$$e^{kt+c} = \frac{P}{1-P}. \quad \text{Then}$$

$$(1-P)\,e^{kt+c} = P \quad \text{and}$$

$$e^{kt+c} = P + Pe^{kt+c} = P\left(1+e^{kt+c}\right)$$

which gives

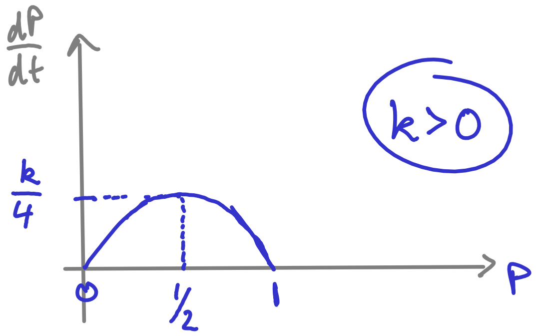
$$P = P(t) = \frac{e^{kt+c}}{1+e^{kt+c}} = \frac{Ae^{kt}}{1+Ae^{kt}}, \quad A = e^{c} = \text{const.}$$

What does this solution look like?

① $0 < P(t) < 1$ for all $t \in \mathbb{R}$

② $\lim_{t \to \infty} P(t) = 1$, $\lim_{t \to -\infty} P(t) = 0$

③ We know $\frac{dP}{dt} = k P(1-P) \begin{cases} > 0 , & k > 0 \\ < 0 , & k < 0 \end{cases}$



④ When is $P(t) = \frac{1}{2} = \frac{Ae^{kt}}{1+Ae^{kt}}$

$$1 + Ae^{kt} = 2Ae^{kt}$$

$$1 = Ae^{kt}$$

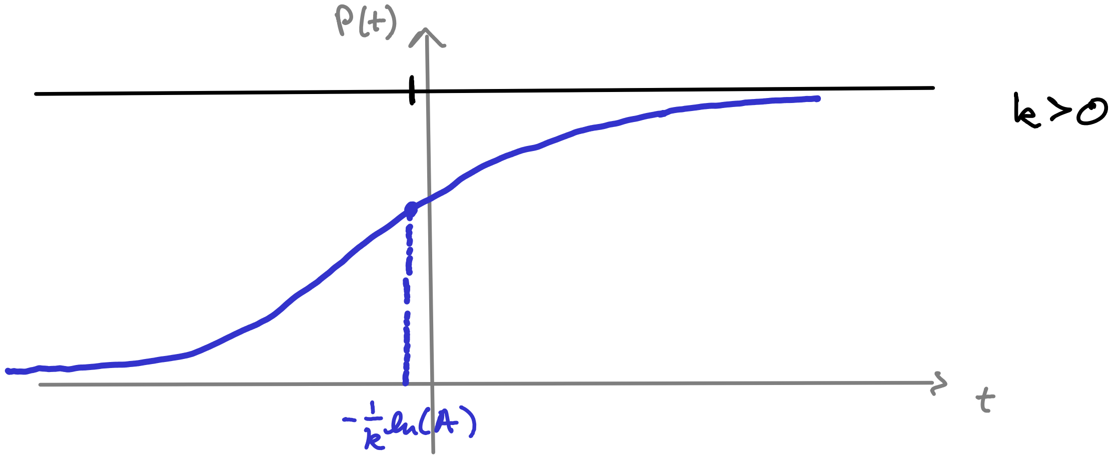
$$\frac{1}{A} = e^{kt}$$

$$\ln\left(\frac{1}{A}\right) = kt \quad , \quad \text{so} \quad t = -\frac{1}{k}\ln(A)$$

⑤ $P(0) = \frac{A}{1+A}$, so A depends on $P(0)$:

$$P(0) = A(1 - P(0)) \quad , \text{i.e.} \quad A = \frac{P(0)}{1-P(0)}$$

⑥ The graph of $P(t)$ is as follows



Half-life

In radioactive decay, the half-life of a material is the time $t_{1/2}$ that it takes to reduce the amount by 50%.

$A(t)$ = amount of radioactive material at time t.

$A(t)$ satisfies $\frac{dA}{dt} = kA$ whose solution is

$$A(t) = A_0 e^{kt}$$

Suppose we measured

t	0	8	16	24	32	40	48
$A(t)$	5.8	5.63	5.5	5.29	5.18	5.05	4.89

From $A(0) = A_0$ we get $A_0 = 5.8$

$A(16) = 5.5 = 5.8\, e^{16k}$. So $k = \frac{1}{16} \ln\left(\frac{5.5}{5.8}\right)$

$= -0.0033$

Half-life then is that $t_{1/2}$ s.t.

$$A(t_{1/2}) = \frac{1}{2} A(0) = \frac{1}{2} A_0 \quad , \quad \text{i.e.}$$

$$e^{k t_{1/2}} = \frac{1}{2} \quad \text{or} \quad t_{1/2} = -\frac{1}{k} \ln(2)$$

MA161 Calculus Tues. 12th March

The Logistic Equation : Based on Stewart (Pg. 595-597) 6th Ed.

Models for Population Growth.

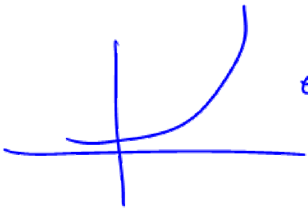
Background. Have studied

$$\frac{dP}{dt} = kP$$

P = population ; $P(t)$ = pop. at time t

k = constant "Little k"

$k > 0$



exponential growth

Logistic equation:

Let K = carrying capacity – max. population that the environment can sustain

We want

- $\frac{dP}{dt} \approx kP$ when P is small
- growth rate to decrease as P grows
- growth rate to become negative if P exceeds K

Model : $$\frac{1}{P}\frac{dP}{dt} = k\left(1 - \frac{P}{K}\right)$$

Note : $\frac{dP}{dt} = kP \iff \frac{1}{P}\frac{dP}{dt} = k$

- when $P << K$ then $1 - \frac{P}{K} \approx 1$: recover original model
- when $P = K$, RHS $= 0 \Rightarrow$ Pop. is constant
- when $P > K$, RHS $= k\left(1 - \frac{P}{K}\right)$ is negative ; population decays

Solving the logistic eqn. (Find $P(t)$ explicitly).

$$\frac{1}{P}\frac{dP}{dt} = k\left(1-\frac{P}{K}\right) \Rightarrow \frac{dP}{dt} = kP\left(1-\frac{P}{K}\right)$$

is a separable equation

$$\Rightarrow \frac{dP}{P\left(1-\frac{P}{K}\right)} = k\,dt$$

$$\Rightarrow \int \frac{dP}{P\left(1-\frac{P}{K}\right)} = \int k\,dt$$

[Aside: Technique of partial fractions

Note: $\int \frac{1}{x}\,dx = \ln|x| + C$; $\int \frac{1}{x-5}\,dx = \ln|x-5| + C$

$$\int \frac{dx}{5x} = \int \frac{1}{5x}\,dx = \frac{1}{5}\int \frac{1}{x}\,dx = \frac{1}{5}\ln|x| + C$$

$$\int \frac{1}{x^2}\cdot dx = \frac{x^{-1}}{-1} + C = \frac{-1}{x} + C$$ Not Logarithmic.

Partial fractions: $\int \frac{3}{x^2-4x}\,dx$

Rewrite $\frac{3}{x^2-4x} = \frac{3}{x(x-4)}$

Let $\frac{3}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4}$ for some A, B and for $\underline{\underline{\text{all}}}$ x

$$= \frac{A(x-4)+Bx}{x(x-4)}$$

$\Rightarrow \quad 3 = A(x-4) + Bx$ for all x

Let $x=4 \Rightarrow 3 = B(4) \Rightarrow B = 3/4$

Let $x=0 \Rightarrow 3 = A(-4) \Rightarrow A = -3/4$

Now $\int \frac{3}{x(x-4)}\,dx = \int \frac{-3/4}{x}\,dx + \int \frac{3/4}{x-4}\,dx$

$$= \frac{-3}{4}\int \frac{1}{x}dx + \frac{3}{4}\int \frac{1}{x-4}dx$$

$$\therefore \int \frac{3}{x^2-4x}dx = \frac{-3}{4}\ln|x| + \frac{3}{4}\ln|x-4| + C$$

Now, $\displaystyle\int \frac{dP}{P\left(1-\frac{P}{K}\right)} = \int kt$

$$= kt + C_1$$

LHS : write $\displaystyle\frac{1}{P\left(1-\frac{P}{K}\right)} = \frac{A}{P} + \frac{B}{1-\frac{P}{K}}$

$$= \frac{A\left(1-\frac{P}{K}\right) + BP}{P\left(1-\frac{P}{K}\right)} \quad \text{for all } P$$

$$\Rightarrow \quad 1 = A\left(1-\frac{P}{K}\right) + BP \quad \text{for all } P$$

Let $P=0 \Rightarrow 1 = A$

Let $P=K \Rightarrow 1 = A(0) + B(K) \Rightarrow B = \frac{1}{K}$

So $\displaystyle\int \frac{1}{P\left(1-\frac{P}{K}\right)}dP = \int \frac{1}{P}dP + \int \frac{1/K}{1-P/K}dP$

$$= \ln|P| + \frac{1}{K}\int \frac{1}{1-P/K}dP$$

Let $x = 1 - P/K \Rightarrow dx = -\frac{dP}{K}$

& $\displaystyle\frac{1}{K}\int \frac{1}{1-P/K}dP = \frac{1}{K}\int \frac{1}{x}\cdot(-K\,dx)$

$$= -1\int \frac{1}{x}dx$$

$$= -\ln|x|$$

$$= -\ln\left|1-\frac{P}{K}\right|$$

$$\therefore \quad \ln|P| - \ln\left|1 - \frac{P}{K}\right| = kt + C$$

$$\Rightarrow \quad \ln\left|\frac{P}{1 - P/K}\right| = kt + C$$

$\Rightarrow$ exp of both sides

$$\Rightarrow \quad \frac{P}{1 - P/K} = e^{kt + C}$$

$$\Rightarrow \quad P = \left(1 - \frac{P}{K}\right) e^{kt + C}$$

$$\Rightarrow \quad P = e^{kt + C} - \frac{P}{K} e^{kt + C}$$

$$\Rightarrow \quad P + \frac{P}{K} e^{kt + C} = e^{kt + C}$$

$$\Rightarrow \quad P\left(1 + \frac{1}{K} e^{kt + C}\right) = e^{kt + C}$$

$$\Rightarrow \quad P = \frac{e^{kt + C}}{1 + \frac{e^{kt + C}}{K}}$$

$$\Rightarrow \quad P = \frac{e^{kt} . e^{C}}{\frac{K + e^{kt} . e^{C}}{K}}$$

$$= \frac{K e^{kt} e^{C}}{K + e^{kt} e^{C}}$$

$$P(t) = \frac{K}{1 + Ae^{-kt}}$$

~~div~~ multiply above & below by e^{-kt}

sleight of hand

$A =$ accumulation of constants

Webwork Problems Sheet 3

Problem 4: "grows at a rate proportional to its size"

mean $\frac{dN}{dt} = kN$ and the solution, $N = N(t) = \#$ bacteria

at time t, is $N(t) = Ae^{kt}$

Know: $N(0) = 700 = A$

$N(7) = 7000 = Ae^{7k} = 700e^{7k}$

So $10 = e^{7k}$ and $\ln(10) = 7k$ which gives

$k = \frac{\ln(10)}{7}$ and hence

$$N(t) = 700\, e^{\frac{\ln(10)}{7} t}$$

$$N(8) = 700 \times 10^{8/7} = 9726.47$$

growth rate after 8 hours is $\frac{dN}{dt}(8) = kN(8)$

$$= \frac{\ln(10)}{7} \times 700 \times 10^{8/7}$$

$$= 3199.4$$

Solve $N(t) = 30000$ for t:

$$700\, e^{\frac{\ln(10)}{7} t} = 30000$$

$$e^{\frac{\ln(10)}{7} t} = \frac{300}{7} \implies \frac{\ln(10)}{7} t = \ln\left(\frac{300}{7}\right)$$

So $t = 7 \dfrac{\ln(300/7)}{\ln(10)} = 11.42$

Problem 6 : $\frac{dy}{dx} = (y-1)(y+1)$

Separate variables and integrate

$$\int \frac{1}{(y-1)(y+1)}\,dy = \int dx = x + c$$

$$\parallel$$

$$\int \left(\frac{1/2}{y-1} + \frac{-1/2}{y+1}\right) dy$$

$$= \frac{1}{2}\ln|y-1| - \frac{1}{2}\ln|y+1|$$

$$= \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right|$$

$$\frac{A}{y-1} + \frac{B}{y+1} = \frac{A(y+1)+B(y-1)}{(y-1)(y+1)}$$

$$= \frac{y(A+B) + A - B}{(y-1)(y+1)}$$

$A+B=0 \Rightarrow B=-A$

$A-B=1 \Longrightarrow 2A=1$

$A=\frac{1}{2}$, $B=-\frac{1}{2}$

Now we have

$$\ln\left|\frac{y-1}{y+1}\right| = 2x+2c$$

$$\frac{y-1}{y+1} = e^{2x+2c} = e^{2c}e^{2x} = Ae^{2x}$$

This gives

$$y-1 = yAe^{2x} + Ae^{2x}$$

$$y(1-Ae^{2x}) = Ae^{2x}+1$$

$$\boxed{y = y(x) = \frac{Ae^{2x}+1}{1-Ae^{2x}}}$$

$$\frac{dy}{dx} = \frac{2Ae^{2x}(1-Ae^{2x}) + 2Ae^{2x}(1+Ae^{2x})}{(1-Ae^{2x})^2}$$

$$= \frac{4Ae^{2x}}{(1-Ae^{2x})^2}$$

$\parallel$ ✓

$$(y-1)(y+1) = y^2-1 = \frac{(Ae^{2x}+1)^2}{(1-Ae^{2x})^2} - 1 = \frac{\cancel{(Ae^{2x})^2} + 2Ae^{2x} + \cancel{1} - (\cancel{1} - 2Ae^{2x} + \cancel{(Ae^{2x})^2})}{(1-Ae^{2x})^2}$$

So $y(3) = \dfrac{Ae^6 + 1}{1 - Ae^6}$ which we want be 0 in order for the graph to pass through $(3,0)$.

This gives $Ae^6 + 1 = 0$, i.e $A = \dfrac{-1}{e^6} = -0.0025$

$$= -e^{-6}$$

$$y(x) = \frac{-e^{-6}e^{2x} + 1}{1 + e^{-6}e^{2x}} = \frac{1 - e^{2x-6}}{1 + e^{2x-6}}$$

More Webwork Problems

Problem 9: The general logistic equation is

$$\frac{dP}{dt} = kP(1-P) \quad \text{where } P \text{ is the ratio of}$$

$$\frac{\text{current population}}{\text{maximal population}}, \quad \text{so} \quad 0 \le P \le 1$$

If we want to measure absolute popupation, say N, then then equation becomes

$$\frac{dN}{dt} = kN\left(1 - \frac{N}{K}\right), \quad \text{where } K \text{ is capacity.}$$

From the problem text we get $k=2$, $K=7750$

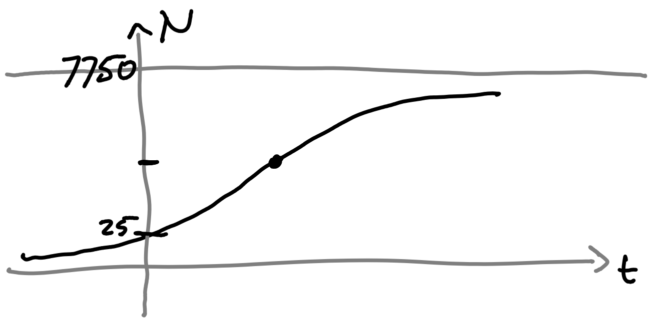
and $N(0) = 25$.

From lecture 20 we have that $N(t) = \dfrac{K}{1+Ae^{-kt}}$.

In the problem this is $N(t) = \dfrac{7750}{1+Ae^{-2t}}$.

Now $N(0) = \dfrac{7750}{1+A} = 25 \Rightarrow \dfrac{7725}{25} = A = 309$

and we get $N(t) = \dfrac{7750}{1+309e^{-2t}}$



The point of inflection is where $\dfrac{dN}{dt}$ has a maximum

$\dfrac{dN}{dt} = 2N\left(1 - \dfrac{N}{7750}\right)$ which is quadratic equation in N

This a parabola whose max is in the middle between its zeros which are a $N=0$ and $N=7750$. This means that at $N=\frac{7750}{2}=3875$ the rate of infection starts to decrease.

Problem 3: The sentence "the coffee cools at $3°C$ per minute when the coffee is $70°C$" means

$$\underset{\text{cools}}{-3} = k(70-22). \quad \text{So } k = \frac{-3}{48} = -\frac{1}{16}$$

Finding the general solution of

$$\frac{dT}{dt} = k(T-A), \qquad A = \text{const.}$$

Put $S = T-A$. Then $\frac{dS}{dt} = \frac{dT}{dt} = kS$, and we know that

$$S(t) = Ce^{kt}, \quad \text{for some constant } C.$$

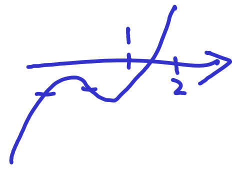
$$\overset{\|}{T(t)-A}$$

and we find $T(t) = A + Ce^{kt}$.

Revision

Newton's Method for approximating zeros:

Example: Find the zero of



$$p(x) = x^3 + x^2 - 3$$

$p(0) = -3$
$p(-1) = -3$
$p(1) = -1$
$p(2) = 9$

$$x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)}$$ Choose $x_0 = 1$.

$$p'(x) = 3x^2 + 2x$$

$$x_{n+1} = x_n - \frac{x_n^3 + x_n^2 - 3}{3x_n^2 + 2x_n}$$

So $x_1 = 1 - \frac{1+1-3}{3+2} = 1 + \frac{1}{5} = \frac{6}{5} = 1.2$

$x_2 = \frac{6}{5} - \dots = 1.175$

$x_3 = 1.174559954559$

$x_4 = x_5 = x_6 = \dots = 1.17455941029$

Integration by Substitution

Find $\int 2x\,(x^2+3)^{50}\,dx$

Put $x^2+3 = u$. So $\frac{du}{dx} = 2x$ or $du = 2x\,dx$ and

$$\int 2x\,(x^2+3)^{50}dx = \int u^{50}\,du = \frac{1}{51}u^{51} + c = \frac{1}{51}(x^2+3)^{51} + c.$$

Similarly $\int (x+2)\sin(x^2+4x-1)\,dx$

Put $x^2+4x-1=u$, So $du=(2x+4)\,dx$

$$=2(x+2)\,dx$$

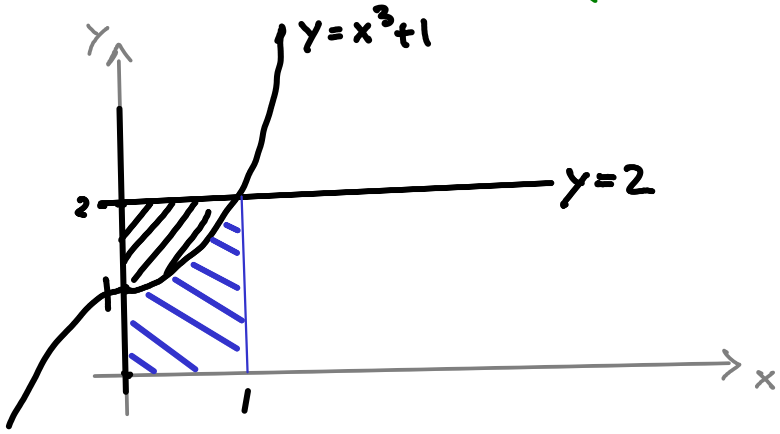
and $\frac{1}{2}du=(x+2)\,dx$. Then

$$\int (x+2)\sin(x^2+4x-1)\,dx = \int \tfrac{1}{2}\sin(u)\,du$$

$$= -\tfrac{1}{2}\cos(u) + c = -\tfrac{1}{2}\cos(x^2+4x-1)+c$$

Volume of revolution

Sketch the area bounded by $y=x^3+1$, $y=2$ and the y-axis and calculate the volume of the solid obtained by rotating it around the x-axis.



The volume is

$$\int_0^1 \pi\left(\underbrace{2^2}_{\text{upper bound}} - \underbrace{(x^3+1)^2}_{\text{lower bound}}\right)dx = \underbrace{\pi\int_0^1 4\,dx}_{\text{disc of rad. 2 and height 1}} - \underbrace{\pi\int_0^1 (x^3+1)^2\,dx}_{\text{blue area rotated.}}$$

$$= \pi \left[4x\right]_0^1 - \pi \int_0^1 \left(x^6 + 2x^3 + 1\right) dx$$

$$= 4\pi - \pi \left[\frac{1}{7}x^7 + \frac{1}{2}x^4 + x\right]_0^1$$

$$= 4\pi - \pi\left(1 + \frac{1}{2} + \frac{1}{7}\right) = 4\pi - \pi\left(\frac{14 + 7 + 2}{14}\right)$$

$$= \pi\left(4 - \frac{23}{14}\right) = \frac{33}{14}\pi$$