Webwork Problems Sheet 3

Problem 4: "grows at a rate proportional to its size"

mean $\frac{dN}{dt} = kN$ and the solution, $N = N(t) = \#$ bacteria at time t, is $N(t) = Ae^{kt}$

Know: $N(0) = 700 = A$

$N(7) = 7000 = Ae^{7k} = 700e^{7k}$

So $10 = e^{7k}$ and $\ln(10) = 7k$ which gives

$k = \frac{\ln(10)}{7}$ and hence

$$N(t) = 700 e^{\frac{\ln(10)}{7} t}$$

$$N(8) = 700 \times 10^{8/7} = 9726.47$$

growth rate after 8 hours is $\frac{dN}{dt}(8) = kN(8)$

$$= \frac{\ln(10)}{7} \times 700 \times 10^{8/7}$$

$$= 3199.4$$

Solve $N(t) = 30000$ for t:

$$700 e^{\frac{\ln(10)}{7} t} = 30000$$

$$e^{\frac{\ln(10)}{7} t} = \frac{300}{7} \implies \frac{\ln(10)}{7} t = \ln\left(\frac{300}{7}\right)$$

So $t = 7 \frac{\ln(300/7)}{\ln(10)} = 11.42$

Problem 6: $\dfrac{dy}{dx} = (y-1)(y+1)$

Separate variables and integrate

$$\int \frac{1}{(y-1)(y+1)}\,dy = \int dx = x + c$$

$$\int \left(\frac{1/2}{y-1} + \frac{-1/2}{y+1} \right) dy$$

$$= \frac{1}{2}\ln|y-1| - \frac{1}{2}\ln|y+1|$$

$$= \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right|$$

$$\frac{A}{y-1} + \frac{B}{y+1} = \frac{A(y+1)+B(y-1)}{(y-1)(y+1)}$$

$$= \frac{y(A+B)+A-B}{(y-1)(y+1)}$$

$A+B=0 \Rightarrow B=-A$

$A-B=1 \Longrightarrow 2A=1$

$A=\frac{1}{2},\ B=-\frac{1}{2}$

Now we have

$$\ln\left|\frac{y-1}{y+1}\right| = 2x+2c$$

$$\frac{y-1}{y+1} = e^{2x+2c} = e^{2c}e^{2x} = Ae^{2x}$$

This gives

$$y-1 = yAe^{2x} + Ae^{2x}$$

$$y(1-Ae^{2x}) = Ae^{2x}+1$$

$$\boxed{y = y(x) = \frac{Ae^{2x}+1}{1-Ae^{2x}}}$$

$$\frac{dy}{dx} = \frac{2Ae^{2x}(1-Ae^{2x}) + 2Ae^{2x}(1+Ae^{2x})}{(1-Ae^{2x})^2}$$

$$= \frac{4Ae^{2x}}{(1-Ae^{2x})^2}$$

$\checkmark$

$$(y-1)(y+1) = y^2-1 = \frac{(Ae^{2x}+1)^2}{(1-Ae^{2x})^2} - 1 = \frac{\cancel{(Ae^{2x})^2} + 2Ae^{2x} + \cancel{1} - (\cancel{1} - 2Ae^{2x} + \cancel{(Ae^{2x})^2})}{(1-Ae^{2x})^2}$$

So $y(3) = \dfrac{Ae^6+1}{1-Ae^6}$ which we want be 0 in order for the graph to pass through $(3,0)$.

This gives $Ae^6+1 = 0$, i.e $A = \dfrac{-1}{e^6} = -0.0025$

$= -e^{-6}$

$$y(x) = \frac{-e^{-6}e^{2x}+1}{1+e^{-6}e^{2x}} = \frac{1-e^{2x-6}}{1+e^{2x-6}}$$