MA161 Calculus Tues. 12^{th} March

The Logistic Equation : Based on Stewart (Pg. 595-597) 6th Ed.

Models for Population Growth.

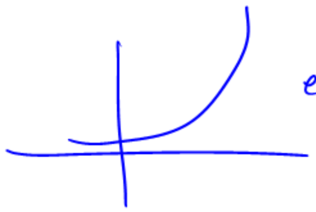
Background. Have studied

$$\frac{dP}{dt} = kP$$

P = population ; $P(t)$ = pop. at time t

k = constant "Little k"

$k > 0$



exponential growth

Logistic equation:

Let K = carrying capacity — max. population that the environment can sustain

We want

- $\frac{dP}{dt} \approx kP$ when P is small
- growth rate to decrease as P grows
- growth rate to become negative if P exceeds K

Model : $$\frac{1}{P}\frac{dP}{dt} = k\left(1 - \frac{P}{K}\right)$$

[Note : $\frac{dP}{dt} = kP \iff \frac{1}{P}\frac{dP}{dt} = k$]

- when $P << K$ then $1 - \frac{P}{K} \approx 1$: recover original model
- when $P = K$, RHS $= 0 \Rightarrow$ Pop. is constant
- when $P > K$, RHS $= k\left(1 - \frac{P}{K}\right)$ is negative ; population decays

Solving the logistic eqn. (Find $P(t)$ explicitly).

$$\frac{1}{P}\frac{dP}{dt} = k\left(1-\frac{P}{K}\right) \Rightarrow \frac{dP}{dt} = kP\left(1-\frac{P}{K}\right)$$

is a separable equation

$$\Rightarrow \frac{dP}{P\left(1-\frac{P}{K}\right)} = k\,dt$$

$$\Rightarrow \int \frac{dP}{P\left(1-\frac{P}{K}\right)} = \int k\,dt$$

[Aside: Technique of partial fractions

Note: $\int \frac{1}{x}\,dx = \ln|x| + C$; $\int \frac{1}{x-5}\,dx = \ln|x-5| + C$

$$\int \frac{dx}{5x} = \int \frac{1}{5x}dx = \frac{1}{5}\int \frac{1}{x}dx = \frac{1}{5}\ln|x| + C$$

$$\int \frac{1}{x^2}\cdot dx = \frac{x^{-1}}{-1} + C = \frac{-1}{x} + C$$ Not Logarithmic.

Partial fractions: $\int \frac{3}{x^2-4x}\,dx$

Rewrite $\frac{3}{x^2-4x} = \frac{3}{x(x-4)}$

Let $\frac{3}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4}$ for some A, B and for $\underline{\underline{\text{all}}}$ x

$$= \frac{A(x-4)+Bx}{x(x-4)}$$

$\Rightarrow 3 = A(x-4) + Bx$ for all x

Let $x=4 \Rightarrow 3 = B(4) \Rightarrow B = 3/4$

Let $x=0 \Rightarrow 3 = A(-4) \Rightarrow A = -3/4$

Now $\int \frac{3}{x(x-4)}dx = \int \frac{-3/4}{x}dx + \int \frac{3/4}{x-4}dx$

$$= \frac{-3}{4}\int \frac{1}{x}dx + \frac{3}{4}\int \frac{1}{x-4}dx$$

$$\therefore \int \frac{3}{x^2-4x}dx = \frac{-3}{4}\ln|x| + \frac{3}{4}\ln|x-4| + C$$

Now, $\displaystyle\int \frac{dP}{P\left(1-\frac{P}{K}\right)} = \int kt$

$$= kt + C_1$$

LHS : write $\displaystyle\frac{1}{P\left(1-\frac{P}{K}\right)} = \frac{A}{P} + \frac{B}{1-\frac{P}{K}}$

$$= \frac{A\left(1-\frac{P}{K}\right) + BP}{P\left(1-\frac{P}{K}\right)} \quad \text{for all } P$$

$$\Rightarrow \quad 1 = A\left(1-\frac{P}{K}\right) + BP \quad \text{for all } P$$

Let $P=0 \Rightarrow 1 = A$

Let $P=K \Rightarrow 1 = A(0) + B(K) \Rightarrow B = \frac{1}{K}$

So $\displaystyle\int \frac{1}{P\left(1-\frac{P}{K}\right)}dP = \int \frac{1}{P}dP + \int \frac{1/K}{1-P/K}dP$

$$= \ln|P| + \frac{1}{K}\int \frac{1}{1-P/K}dP$$

Let $x = 1-P/K \Rightarrow dx = -\frac{dP}{K}$

& $\displaystyle\frac{1}{K}\int \frac{1}{1-P/K}dP = \frac{1}{K}\int \frac{1}{x}\cdot(-K\,dx)$

$$= -1\int \frac{1}{x}dx$$

$$= -\ln|x|$$

$$= -\ln\left|1-\frac{P}{K}\right|$$

$$\therefore \quad \ln|P| - \ln\left|1 - \frac{P}{K}\right| = kt + C$$

$$\Rightarrow \quad \ln\left|\frac{P}{1 - P/K}\right| = kt + C$$

$\Rightarrow$ exp of both sides

$$\Rightarrow \quad \frac{P}{1 - P/K} = e^{kt + C}$$

$$\Rightarrow \quad P = \left(1 - \frac{P}{K}\right) e^{kt + C}$$

$$\Rightarrow \quad P = e^{kt + C} - \frac{P}{K} e^{kt + C}$$

$$\Rightarrow \quad P + \frac{P}{K} e^{kt + C} = e^{kt + C}$$

$$\Rightarrow \quad P\left(1 + \frac{1}{K} \cdot e^{kt + C}\right) = e^{kt + C}$$

$$\Rightarrow \quad P = \frac{e^{kt + C}}{1 + \frac{e^{kt + C}}{K}}$$

$$\Rightarrow \quad P = \frac{e^{kt} \cdot e^{C}}{\frac{K + e^{kt} \cdot e^{C}}{K}}$$

$$= \frac{K e^{kt} e^{C}}{K + e^{kt} e^{C}}$$

~~div~~ multiply above & below by e^{-kt}

$$P(t) = \frac{K}{1 + Ae^{-kt}}$$

sleight of hand

$A =$ accumulation of constants