The logistic equation solved correctly:

The logistic equation is

$$\frac{dP}{dt} = kP(1-P)$$

We solve it by separating variables

$$\int \frac{1}{P(1-P)}\,dP = \int k\,dt = kt + c$$

$$\parallel$$

$$\int \left(\frac{1}{P} + \frac{1}{1-P}\right) dP = \ln(P) - \ln(1-P)$$

$$= \ln\left(\frac{P}{1-P}\right)$$

So $kt + c = \ln \frac{P}{1-P}$ and we get

$$e^{kt+c} = \frac{P}{1-P}.$$ Then

$$(1-P)e^{kt+c} = P$$ and

$$e^{kt+c} = P + Pe^{kt+c} = P(1+e^{kt+c})$$

which gives

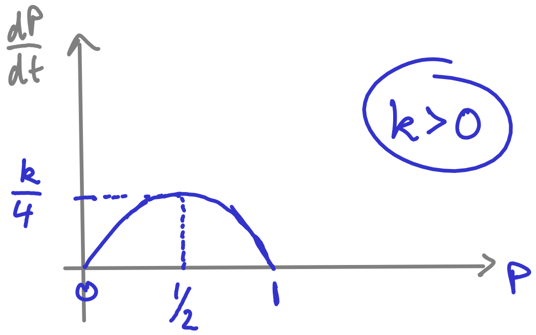
$$P = P(t) = \frac{e^{kt+c}}{1+e^{kt+c}} = \frac{Ae^{kt}}{1+Ae^{kt}}, \quad A = e^{c} = \text{const.}$$

What does this solution look like?

① $0 < P(t) < 1$ for all $t \in \mathbb{R}$

② $\lim_{t\to\infty} P(t) = 1$, $\lim_{t\to-\infty} P(t) = 0$

③ We know $\frac{dP}{dt} = kP(1-P) \begin{cases} >0, & k>0 \\ <0, & k<0 \end{cases}$



④ When is $P(t) = \frac{1}{2} = \frac{Ae^{kt}}{1+Ae^{kt}}$

$$1 + Ae^{kt} = 2Ae^{kt}$$

$$1 = Ae^{kt}$$

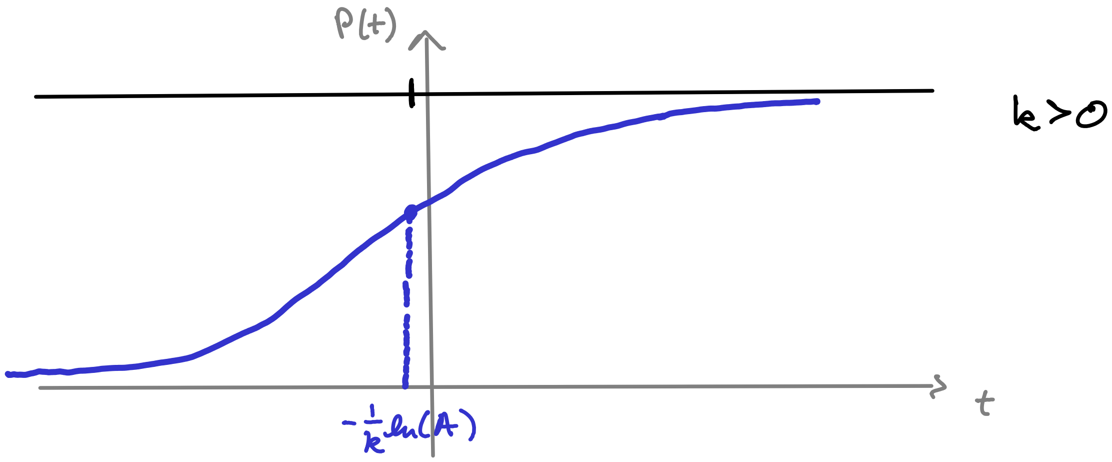
$$\frac{1}{A} = e^{kt}$$

$$\ln\left(\frac{1}{A}\right) = kt, \quad \text{so} \quad t = -\frac{1}{k}\ln(A)$$

⑤ $P(0) = \frac{A}{1+A}$, so A depends on $P(0)$:

$$P(0) = A(1 - P(0)), \text{ i.e. } A = \frac{P(0)}{1-P(0)}$$

⑥ The graph of $P(t)$ is as follows



Half-life

In radioactive decay, the half-life of a material is the time $t_{1/2}$ that it takes to reduce the amount by 50%.

$A(t)$ = amount of radioactive material at time t.

$A(t)$ satisfies $\frac{dA}{dt} = kA$ whose solution is

$$A(t) = A_0 e^{kt}$$

Suppose we measured

t	0	8	16	24	32	40	48
$A(t)$	5.8	5.63	5.5	5.29	5.18	5.05	4.89

From $A(0) = A_0$ we get $A_0 = 5.8$

$A(16) = 5.5 = 5.8\, e^{16k}$. So $k = \frac{1}{16} \ln\left(\frac{5.5}{5.8}\right)$

$= -0.0033$

Half-life then is that $t_{1/2}$ s.t.

$$A(t_{1/2}) = \frac{1}{2} A(0) = \frac{1}{2} A_0 \quad , \text{ i.e.}$$

$$e^{k t_{1/2}} = \frac{1}{2} \quad \text{or} \quad t_{1/2} = -\frac{1}{k} \ln(2)$$