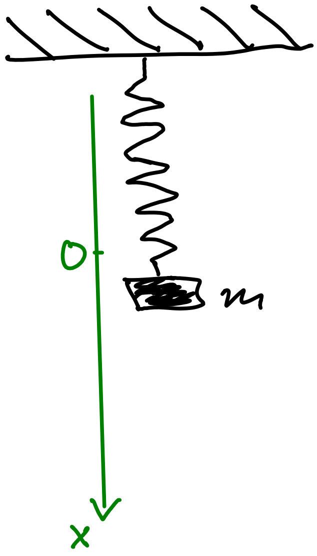
More differential equations



Express the force in two ways.

① Spring constant $k > 0$. The force generated by the spring is $-kx$

② Newton's law says

force = accelleration × mass, from which we get $m\dfrac{d^2x}{dt^2}$

So the motion is governed by $m\dfrac{d^2x}{dt^2} = -kx$, or

$$mx'' + kx = 0 \quad \text{or} \quad x'' = -\frac{k}{m}x.$$

One solution is $x(t) = A\sin\left(\sqrt{\frac{k}{m}}\,t\right)$ and another solution is $x(t) = B\cos\left(\sqrt{\frac{k}{m}}\,t\right)$ and so is

$$x(t) = A\sin\left(\sqrt{\frac{k}{m}}\,t\right) + B\cos\left(\sqrt{\frac{k}{m}}\,t\right)$$

It's a fact that this is the general solution to the differential equation. It is a superposition of two independent solutions which is all a 2^{nd} order diff. equ. can have.

The constants A and B are determined by initial conditions, like initial position and initial velocity, i.e.

$$x(0) = B \quad \text{and} \quad x'(0) = A\sqrt{\frac{k}{m}}$$

More population growth

The simple minded $\frac{dP}{dt} = kP$ is not realistic.

No population will grow arbitrarily large.

A more realistic model assumes a capacity which is an upper bound and measures the proportion of that capacity. For small populations the growth should be exponential. One equation that captures these features is

$$\frac{dP}{dt} = kP(1-P)$$

which is called the <u>logistic equation</u>.

This is a separable equation:

$$\frac{1}{P(1-P)}\,dP = k\,dt$$ and we can integrate

$$\int \frac{1}{P(1-P)}\,dP = \int k\,dt = kt + C$$

$$\|$$

$$\int \left(\frac{1}{P} + \frac{1}{1-P}\right) dP = \ln P \cancel{+} \ln(1-P)$$

$$= \ln(P(1-P))$$

WRONG!
Why?

This gives $P(1-P) = Ae^{kt}$, where $A = e^{C}$.