Differential Equations

A <u>differential equation</u> is an equation involving a function and some of its derivatives.

For example: ① $y' = xy$ where $y = y(x)$

② $y'' + 3y' + y = 0$

A solution of a differential equation is a function satisfying the equation. In general it is difficult to find a solution.

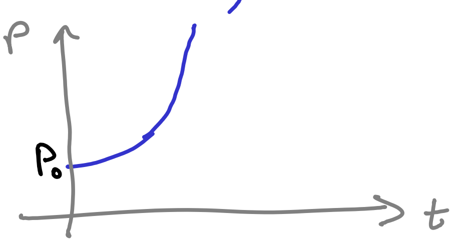
More realistic examples:

① <u>Population growth</u>: Since a species has more offspring the more individuals there are, we could expect the growth rate $\left(\frac{dP}{dt}\right)$ to be proportional to the actual population (P). This yields the differential equation $\frac{dP}{dt} = kP$, for some constant k.

Or $P' = kP$. $k > 0$

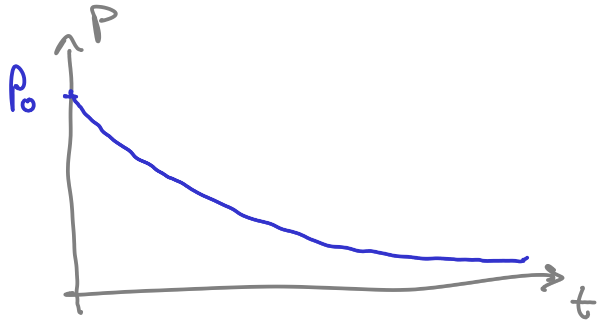
TERMINOLOGY: P is the dependent variable
t is the independent variable

One solution is $P(t) = P_0 e^{kt}$, where $P_0 = P(0)$, the initial population.



Exponential Decay: Most prominently radioactive decay is governed by the equation $\frac{dP}{dt} = kP$, $k<0$.

Again, a solution is $P(t) = P_0 e^{kt}$



How does one come up with a solution??

A differential equation is called separable if it can be rewritten with all occurrences of the dependent variable one side and all occurrences of the independent variable one the other side.

For $\frac{dP}{dt} = kP$ this means that we rewrite it as

$$\frac{1}{P}\,dP = k\,dt$$

and now we can integrate

$$\int \frac{1}{P}\,dP = \int k\,dt$$

and we get

$$\ln|P| = \int k\,dt = kt + C, \quad C = \text{const.}$$

$$P = e^{\ln P} = e^{kt+C} = e^C e^{kt} = \hat{C} e^{kt}$$

Problem: A sample of radioactive material is meassured twice, once at time $t=1$ and at time $t=5$ where it is found that $P(1)=50$ and $P(5)=48$.

① At what time is $P(t)=35$?

② What is $P(20)$?

Solⁿ: We know $P(t) = Ce^{kt}$ so

$$50 = P(1) = Ce^{k}$$
$$48 = P(5) = Ce^{5k}$$

$$\frac{48}{50} = \frac{Ce^{5k}}{Ce^{k}} = e^{4k} \implies 4k = \ln\left(\frac{48}{50}\right)$$

$$k = \frac{1}{4}\ln\left(\frac{48}{50}\right)$$

$$C = \frac{50}{e^{k}} = \frac{50}{\left(\frac{48}{50}\right)^{1/4}}$$

② $$P(20) = \frac{50}{\left(\frac{48}{50}\right)^{1/4}} e^{5\ln\left(\frac{48}{50}\right)} = 50\left(\frac{50}{48}\right)^{1/4}\left(\frac{48}{50}\right)^{5}$$

① Solve $$35 = 50\left(\frac{50}{48}\right)^{1/4} e^{\frac{1}{4}\ln\left(\frac{48}{50}\right)t}$$ for t:

$$\ln\left(\frac{35}{50}\left(\frac{48}{50}\right)^{1/4}\right) = \frac{1}{4}\ln\left(\frac{48}{50}\right)t$$