Homework Question 10 Assignment 2

Consider $\displaystyle\int \frac{2x}{x^2-1}\,dx$.

① Partial Fractions

$$\frac{2x}{x^2-1} = \frac{2x}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$= \frac{A(x+1) + B(x-1)}{x^2-1}$$

$$= \frac{x(A+B) + A - B}{x^2-1}$$

So $2 = A + B$ and $A - B = 0$.

$$\Downarrow$$

$$2 = 2A \quad \Longleftarrow \quad A = B$$

Hence $A = 1$ and $B = 1$.

$$\text{So} \quad \int \frac{2x}{x^2-1}\,dx = \int\left(\frac{1}{x-1} + \frac{1}{x+1}\right)dx$$

$$= \ln|x-1| + \ln|x+1| + c$$

② Substitution: $w = x^2 - 1$. $\quad dw = 2x\,dx$

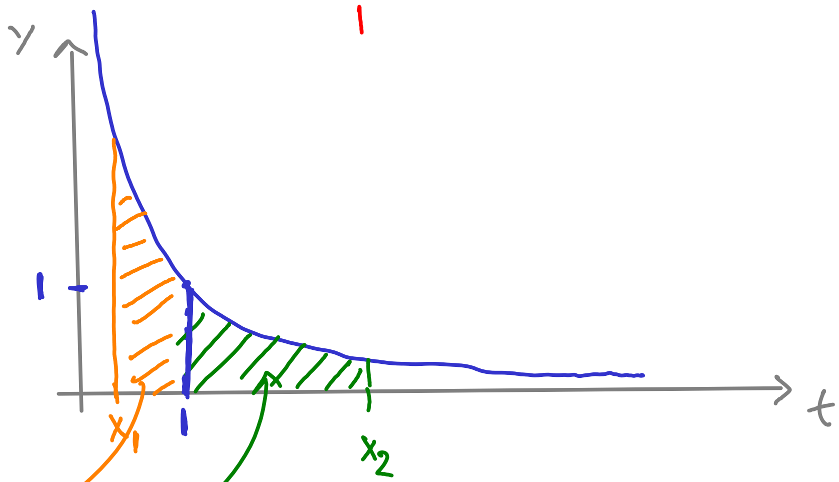
$$\text{So} \quad \int \frac{2x}{x^2-1}\,dx = \int \frac{dw}{w} = \int \frac{1}{w}\,dw$$

$$= \ln|w| + C = \ln|x^2-1| + C$$

Exponential and logarithmic Functions

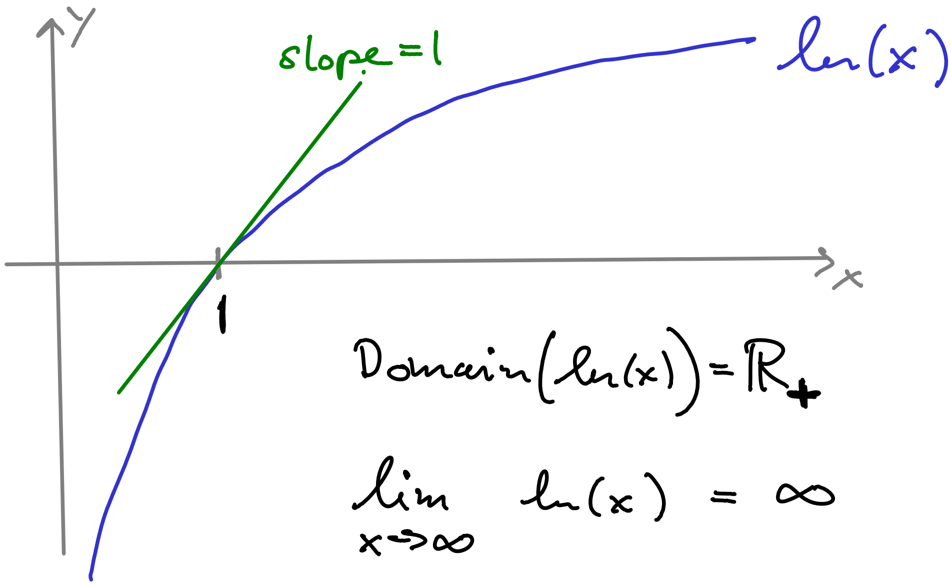
The natural way to define the logarithm function is as

$$\ln(x) = \int_1^x \frac{1}{t}\,dt$$



This area is $\ln(x_2)$

This area is $-\ln(x_1)$



$$\text{Domain}(\ln(x)) = \mathbb{R}_+$$

$$\lim_{x \to \infty} \ln(x) = \infty$$

$$\lim_{x \to 0^+} \ln(x) = -\infty$$

$\ln(x)$ is injective (one-to-one) and hence has an inverse function.

The inverse function of $\ln(x)$ is the exponential function. That means

$$y = \ln(x) \iff e^y = x$$

Also $$e^{\ln(x)} = x \quad \text{and} \quad \ln(e^y) = y$$

<u>Some facts:</u>

$$e^{a+b} = e^a e^b \qquad \ln(pq) = \ln(p) + \ln(q)$$

$$(e^a)^b = e^{ba} \qquad k\ln(p) = \ln(p^k)$$

<u>General exponentials</u>

Interest rate: If you get 3.75% interest every on an initial amount of E_0 euros, then after t years, you have

$$E(t) = E_0 \cdot (1.0375)^t \text{ euros.}$$

This is of the form $E(t) = K a^t$, a, K const.

Problem: What is $\frac{d}{dt}\left(k a^{t}\right)$?

Sol^n: $\frac{d}{dt}(k a^t) = k \frac{d}{dt}(a^t)$

$$= k \frac{d}{dt}\left(e^{\ln(a)}\right)^t \qquad \left[a = e^{\ln(a)}\right]$$

$$= k \frac{d}{dt}\left(e^{\ln(a)t}\right) \qquad \text{chain rule}$$

$$= k\, e^{\ln(a)t} \frac{d}{dt}\left(\ln(a)t\right)$$

$$= k\, e^{\ln(a)t} \ln(a)$$

$$= k \underline{\underline{\ln(a)}}\, a^t$$

To sum up $\frac{d}{dt}\left(k a^t\right) = k \ln(a)\, a^t$.